Supplement for *Measure, Integration & Real Analysis*

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This supplement for *Measure, Integration & Real Analysis* should refresh your understanding of standard definitions, notation, and results from undergraduate real analysis. You will then be ready to read *Measure, Integration & Real Analysis* (click the link for a free, legal, electronic version of the book).

The set $\mathbf{R}$ of real numbers, with the usual operations of addition and multiplication and the usual ordering, is a complete ordered field. This supplement begins by explaining the meaning of the last three words of the previous sentence. Because you have used the ordered field properties of $\mathbf{R}$ since childhood, in this supplement we emphasize the deep properties that follow from completeness.

Section B of this supplement presents a construction of the real numbers using Dedekind cuts (this is the only section of this supplement not used in *Measure, Integration & Real Analysis*, so you can choose to skip this section).

Section C focuses on the crucial concepts of supremum and infimum. The basic properties of open and closed subsets of $\mathbf{R}^n$ are discussed in Section D.

In Section E of this supplement we prove the Bolzano–Weierstrass Theorem, which is then used as a key tool for results concerning uniform continuity and maxima/minima of continuous functions on closed bounded subsets of $\mathbf{R}^n$.

Nineteenth-century painting of Cicero discovering the tomb of Archimedes. In Section C we see how the Archimedean Property of the real numbers follows from the completeness property.

A Complete Ordered Fields

Fields

The algebraic structure of a field captures the arithmetic properties we expect of the real numbers. In the definition of a field below, an operation of addition on a set $\mathbf{F}$ is a function that assigns an element of $\mathbf{F}$ denoted $a+b$ to each ordered pair (a,b) of elements of $\mathbf{F}$. An operation of multiplication on a set $\mathbf{F}$ is a function that assigns an element of $\mathbf{F}$ denoted ab or $a \cdot b$ to each ordered pair (a,b) of elements of $\mathbf{F}$.

0.1 Definition ***field***

A *field* is a set $\mathbf{F}$ along with operations of addition and multiplication on $\mathbf{F}$ that have the following properties:

- **commutativity:** $a+b=b+a$ and $ab=ba$ for all $a,b \in \mathbf{F}$;
- **associativity:** $(a+b)+c = a+(b+c)$ and $(ab)c = a(bc)$ for all $a,b,c \in \mathbf{F}$;
- **distributive property:** $a(b+c) = ab+ac$ for all $a,b,c \in \mathbf{F}$;
- **additive identity:** there exists an element $0 \in \mathbf{F}$ such that $a+0=a$ for all $a \in \mathbf{F}$;
- **additive inverse:** for each $a \in \mathbf{F}$, there exists an element $-a \in \mathbf{F}$ such that $a+(-a)=0$;
- **multiplicative identity:** there exists an element $1 \in \mathbf{F}$ such that $1 \neq 0$ and $a1 = a$ for all $a \in \mathbf{F}$;
- **multiplicative inverse:** for each $a \in \mathbf{F}$ with $a \neq 0$, there exists an element $a^{-1} \in \mathbf{F}$ such that $aa^{-1} = 1$.

The set $\mathbf{Q}$ of rational numbers, with the usual operations of addition and multiplication, is a field. As another example, the set $\{0,1\}$, with the usual operations of addition and multiplication except that $1+1$ is defined to be 0, is a field.

Because 0 is the additive identity in a field, the result below connects addition and multiplication. The only field property that connects addition and multiplication is the distributive property. Thus the proof of the result below must use the distributive property. With that hint, the main idea of the proof (writing 0 as $0+0$ and then using the distributive property) becomes easier to discover.

0.2 $a0 = 0$

Suppose $\mathbf{F}$ is a field. Then $a0 = 0$ for every $a \in \mathbf{F}$.

Proof Suppose $a \in \mathbf{F}$. Then

$$\begin{aligned} a0 &= a(0+0) \\ &= a0 + a0. \end{aligned}$$

Now add $-(a0)$ to each side of the equation above, getting $0 = a0$, as desired. ∎

The familiar properties of arithmetic all follow easily from the field properties listed in the definition of a field. For example, here are a few properties of the additive inverse.

0.3 ***properties of the additive inverse in a field***

Suppose $\mathbf{F}$ is a field. Then

(a) for each $a \in \mathbf{F}$, the additive inverse of a is unique (thus the notation $-a$ makes sense);

(b) $-(-a) = a$ for each $a \in \mathbf{F}$;

(c) $(-1)a = -a$ for each $a \in \mathbf{F}$.

You should be able to write down analogous properties for the multiplicative inverse in a field. The proofs of the easy and familiar field properties are not provided here because we need to get to other topics. However, you may benefit by finding your own proof of the result above and other basic results about the arithmetic of fields.

Subtraction and division are defined in a field using the appropriate inverse, as follows.

0.4 Definition ***subtraction; division***

Suppose $\mathbf{F}$ is a field and $a, b \in \mathbf{F}$.

- The *difference* $a - b$ is defined by the equation

$$a - b = a + (-b).$$

- If $b \neq 0$, then the *quotient* $\frac{a}{b}$ (which is also denoted by a/b and by $a \div b$) is defined by the equation

$$\frac{a}{b} = ab^{-1}.$$

Ordered Fields

The usual arithmetic properties that we expect of the real numbers follow from the definition of a field. However, more structure is needed to generate meaning for the order relationship $a < b$ that we expect of the real numbers. The easiest way to get at order properties in a field comes from designating a subset to be thought of as the positive numbers. Then the ordering $a < b$ can be defined to mean that $b - a$ is positive.

As motivation for the following definition, think of the properties we expect of the positive numbers as a subset of the real numbers: every real number is either positive or 0 or its additive inverse is positive; a real number and its additive inverse cannot both be positive; the sum and product of two positive numbers are both positive numbers.

In the following definition, the symbol P should remind you of the positive numbers.

0.5 Definition ***ordered field; positive***

An *ordered field* is a field $\mathbf{F}$ along with a subset P of $\mathbf{F}$, called the *positive* subset, with the following properties:

- if $a \in \mathbf{F}$, then $a \in P$ or $a = 0$ or $-a \in P$;
- if $a \in P$, then $-a \notin P$;
- if $a, b \in P$, then $a + b \in P$ and $ab \in P$.

For example, the field $\mathbf{Q}$ of rational numbers, with the usual operations of addition and multiplication and with P denoting the usual set of positive rational numbers, is an ordered field.

Because you are already familiar with the properties of the positive numbers, the statements and easy proofs of results concerning the positive subset are left to you as exercises. The following result and its proof give an example of how to work with the definition of an ordered field.

0.6 ***the positive subset is closed under multiplicative inverse***

Suppose $\mathbf{F}$ is an ordered field with positive subset P. Then

(a) $1 \in P$;

(b) $a^{-1} \in P$ for every $a \in P$.

Proof To prove (a), note that the definition of an ordered field implies that either $1 \in P$ or $-1 \in P$. If $1 \in P$, then we are done. If $-1 \in P$, then $1 = (-1)(-1) \in P$ (because P is closed under multiplication). Either way, we conclude that $1 \in P$.

To prove (b), suppose $a \in P$. If we had $-a^{-1} \in P$, then we would have $-1 = (-a^{-1})a \in P$ (because P is closed under multiplication), which contradicts the second bullet point of 0.5. Thus $a^{-1} \in P$, as desired. ■

Now we use the positive subset of an ordered field to define the order relations.

0.7 Definition ***less than; greater than***

Suppose $\mathbf{F}$ is an ordered field with positive subset P. Suppose $a, b \in \mathbf{F}$. Then

- $a < b$ is defined to mean $b - a \in P$;
- $a \leq b$ is defined to mean $a < b$ or $a = b$;
- $a > b$ is defined to mean $b < a$;
- $a \geq b$ is defined to mean $a > b$ or $a = b$.

An important but easy result is proved below to give you an indication of the pattern of proofs involving order properties. However, the other statements and proofs of the simple ordering properties are left to you as exercises because you already have many years of experience with the appropriate properties, and we need to get to other topics.

Notice that the statement $0 < b$ is equivalent to the statement $b \in P$.

0.8 ***transitivity***

Suppose $\mathbf{F}$ is an ordered field and $a, b, c \in \mathbf{F}$. If $a < b$ and $b < c$, then $a < c$.

Proof Suppose $a < b$ and $b < c$. Hence $c - b \in P$ and $b - a \in P$. Because P is closed under addition, we conclude that

$$c - a = (c - b) + (b - a) \in P.$$

Thus $a < c$, as desired. ■

The familiar concept of absolute value can be defined on an ordered field, as follows.

0.9 Definition ***absolute value***

Suppose $\mathbf{F}$ is an ordered field and $b \in \mathbf{F}$. The *absolute value* of b, denoted $|b|$, is defined by

$$|b| = \begin{cases} b & \text{if } b \geq 0, \\ -b & \text{if } b < 0. \end{cases}$$

The observation that $b \leq |b|$ and $-b \leq |b|$ for every b in an ordered field $\mathbf{F}$ provides the key to the proof of our next result.

0.10 $|a + b| \leq |a| + |b|$

Suppose $\mathbf{F}$ is an ordered field and $a, b \in \mathbf{F}$. Then

$$|a + b| \leq |a| + |b|.$$

Proof First suppose $a + b \geq 0$. In that case, we have

$$|a + b| = a + b \leq |a| + |b|.$$

Now suppose $a + b < 0$. In that case, we have

$$|a + b| = -(a + b) = (-a) + (-b) \leq |a| + |b|,$$

completing the proof. ■

The next result allows us to think of $\mathbf{Q}$, the ordered field of rational numbers with the usual operations of addition and multiplication and the usual ordering, as contained in each ordered field.

0.11 *every ordered field contains* Q

Suppose $\mathbf{F}$ is an ordered field. Define $\varphi\colon \mathbf{Q} \to \mathbf{F}$ as follows: $\varphi(0) = 0$, and for m and n positive integers with no common integer factors bigger than 1, let

$$\varphi\left(\tfrac{m}{n}\right) = (\underbrace{1 + \cdots + 1}_{m \text{ times}})(\underbrace{1 + \cdots + 1}_{n \text{ times}})^{-1}$$

and let

$$\varphi\left(-\tfrac{m}{n}\right) = \big(\underbrace{(-1) + \cdots + (-1)}_{m \text{ times}}\big)(\underbrace{1 + \cdots + 1}_{n \text{ times}})^{-1},$$

where each 1 above is the multiplicative identity in $\mathbf{F}$. Then φ is a one-to-one function that preserves all the ordered field properties.

Proof

The properties of an ordered field imply that $\underbrace{1 + \cdots + 1}_{n \text{ times}} > 0$. *In particular,* $\underbrace{1 + \cdots + 1}_{n \text{ times}} \neq 0$, *and thus the multiplicative inverses above make sense in* $\mathbf{F}$.

To show that φ is a one-to-one function, suppose first that

$$\varphi\left(\tfrac{m}{n}\right) = \varphi\left(\tfrac{p}{q}\right),$$

where m, n, p, q are positive integers and both fractions are in reduced form. The equality above implies that

$$(\underbrace{1 + \cdots + 1}_{m \text{ times}})(\underbrace{1 + \cdots + 1}_{q \text{ times}}) = (\underbrace{1 + \cdots + 1}_{p \text{ times}})(\underbrace{1 + \cdots + 1}_{n \text{ times}}).$$

Repeated applications of the distributive property show that both sides of the equation above are a sum, with 1 appearing mq times on the left side and pn times on the right side. Thus

$$mq = pn$$

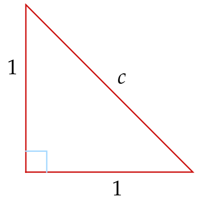
(because otherwise, after adding the additive inverse of the shorter side to both sides of the equation above, we would have a sum of 1's equaling 0, which would violate the order properties). Thus $\frac{m}{n} = \frac{p}{q}$, which shows that the restriction of φ to the positive rational numbers is a one-to-one function.

Using the ideas of the paragraph above, the reader should be able to show that φ is a one-to-one function on all of $\mathbf{Q}$. The reader should also be able to verify that φ preserves all the ordered field properties. In other words, $\varphi(a + b) = \varphi(a) + \varphi(b)$, $\varphi(ab) = \varphi(a)\varphi(b)$, $\varphi(-a) = -\varphi(a)$, $\varphi(a^{-1}) = \varphi(a)^{-1}$, and $\varphi(a) > 0$ if and only if $a > 0$ for all $a, b \in \mathbf{Q}$ (with $a \neq 0$ for the multiplicative inverse condition). ■

The result above means that we can identify $a \in \mathbf{Q}$ with $\varphi(a) \in \mathbf{F}$. Thus from now on we think of $\mathbf{Q}$ as a subset of each ordered field.

Completeness

The Pythagorean Theorem implies that a right triangle with two legs of length 1 has a hypotenuse whose length c satisfies the equation $c^2 = 2$. The ancient Greeks discovered that the rational numbers are not rich enough to have such a number, as shown by the next result.



An isosceles right triangle, with $c^2 = 2$.

0.12 *no rational number has a square equal to* 2

There does not exist a rational number whose square is 2.

Proof Suppose there exist integers m and n such that

$$\left(\frac{m}{n}\right)^2 = 2.$$

By canceling common factors, we can choose m and n to have no common integer factors greater than 1. In other words, we can assume that $\frac{m}{n}$ is a fraction in reduced form.

The equation above is equivalent to the equation

$$m^2 = 2n^2.$$

Thus m^2 is even. Hence m is even. Thus $m = 2k$ for some integer k. Substituting $2k$ for m in the equation above gives

$$4k^2 = 2n^2,$$

or equivalently

$$2k^2 = n^2.$$

Thus n^2 is even. Hence n is even.

We have now shown that both m and n are even, contradicting our choice of m and n as having no common integer factors greater than 1.

This contradiction means our original assumption that there is a rational number whose square equals 2 was incorrect, completing the proof. ■

Pythagoras explaining his work (from The School of Athens*, painted by Raphael around 1510).*

"When you have excluded the impossible, whatever remains, however improbable, must be the truth."
—Sherlock Holmes

Intuitively, we expect that the length of any line segment (including the hypotenuse of a right triangle with two legs of length 1) should be a real number. Thus the result above tells us that there should be a real number $\sqrt{2}$ that is not rational. The rational numbers $\mathbf{Q}$, with the usual operations of addition and multiplication, form an ordered field. Hence we see that we need more than the properties of an ordered field to describe our notion of the real numbers.

Experience has shown that the best way to capture the expected properties of the real numbers comes through consideration of upper bounds.

0.13 Definition ***upper bound***

Suppose $\mathbf{F}$ is an ordered field and $A \subseteq \mathbf{F}$. An element b of $\mathbf{F}$ is called an *upper bound* of A if $a \leq b$ for every $a \in A$.

0.14 Example *upper bounds*

If we work in the ordered field $\mathbf{Q}$ of rational numbers and

$$A_1 = \{a \in \mathbf{Q} : a \leq 3\} \quad \text{and} \quad A_2 = \{a \in \mathbf{Q} : a < 3\},$$

then every rational number b with $b \geq 3$ is an upper bound of A_1 and of A_2.

Some subsets of $\mathbf{Q}$ do not have an upper bound.

0.15 Example *no upper bound*

If we work in the ordered field $\mathbf{Q}$ of rational numbers and A is the set of even integers, then A does not have an upper bound.

Now we come to a crucial definition.

0.16 Definition ***least upper bound***

Suppose $\mathbf{F}$ is an ordered field and $A \subseteq \mathbf{F}$. An element b of $\mathbf{F}$ is called a *least upper bound* of A if both the following conditions hold:

- b is an upper bound of A;
- $b \leq c$ for every upper bound c of A.

In other words, a least upper bound of a set is an upper bound that is less than or equal to all the other upper bounds of the set.

If b and d are both least upper bounds of a subset A of an ordered field $\mathbf{F}$, then $b \leq d$ and $d \leq b$ (by the second bullet point above), and hence $b = d$. In other words, a least upper bound of a set, if it exists, is unique.

0.17 Example *least upper bounds*

If we work in the ordered field $\mathbf{Q}$ of rational numbers and

$$A_1 = \{a \in \mathbf{Q} : a \leq 3\} \quad \text{and} \quad A_2 = \{a \in \mathbf{Q} : a < 3\},$$

then 3 is the least upper bound of A_1 and 3 is the least upper bound of A_2. Note that the least upper bound 3 of A_1 is an element of A_1, but the least upper bound 3 of A_2 is not an element of A_2.

The next example shows that a nonempty subset of an ordered field can have an upper bound without having a least upper bound.

0.18 Example *$\{a \in \mathbf{Q} : a^2 < 2\}$ does not have a least upper bound in $\mathbf{Q}$*

Suppose $b \in \mathbf{Q}$. We want to show that b is not a least upper bound of $\{a \in \mathbf{Q} : a^2 < 2\}$. We know from 0.12 that $b^2 \neq 2$. Thus $b^2 < 2$ or $b^2 > 2$.

Intuitively, the least upper bound of $\{a \in \mathbf{Q} : a^2 < 2\}$ should be $\sqrt{2}$, but there is no such number in $\mathbf{Q}$.

First consider the case where $b^2 < 2$. If we can find a positive rational number δ such that $(b + \delta)^2 < 2$, then b is not an upper bound of $\{a \in \mathbf{Q} : a^2 < 2\}$. Take

The idea here is that if $b^2 < 2$, then we can find a number slightly bigger than b in $\{a \in \mathbf{Q} : a^2 < 2\}$.

$$\delta = \frac{2 - b^2}{5}.$$

Because $b < 2$ and $0 < \delta < 1$, we have $2b + \delta < 5$ and

$$\begin{aligned} (b + \delta)^2 &= b^2 + (2b + \delta)\delta \\ &< b^2 + 5\delta \\ &= 2. \end{aligned}$$

Thus b is not an upper bound of $\{a \in \mathbf{Q} : a^2 < 2\}$.

Now consider the case where $b > 0$ and $b^2 > 2$. If we can find a rational number δ such that $0 < \delta < b$ and $(b - \delta)^2 > 2$, then $b - \delta$ is an upper bound of $\{a \in \mathbf{Q} : a^2 < 2\}$, which implies that b is not a least upper bound of $\{a \in \mathbf{Q} : a^2 < 2\}$. Take

The idea here is that if $b^2 > 2$, then we can find a number slightly smaller than b that is an upper bound of $\{a \in \mathbf{Q} : a^2 < 2\}$.

$$\delta = \frac{b^2 - 2}{2b}.$$

Then $0 < \delta < b$ and

$$\begin{aligned} (b - \delta)^2 &= b^2 - 2b\delta + \delta^2 \\ &> b^2 - 2b\delta \\ &= 2. \end{aligned}$$

Thus b is not a least upper bound of $\{a \in \mathbf{Q} : a^2 < 2\}$, which completes the explanation of why $\{a \in \mathbf{Q} : a^2 < 2\}$ does not have a least upper bound in $\mathbf{Q}$.

The last example indicates that the absence of a least upper bound prevents the field of rational numbers from having a square root of 2, motivating the next definition.

0.19 Definition *complete ordered field*

An ordered field $\mathbf{F}$ is called *complete* if every nonempty subset of $\mathbf{F}$ that has an upper bound has a least upper bound.

Example 0.18 shows that $\mathbf{Q}$ is not a complete ordered field. Our intuitive notion of the real line as having no holes indicates that the field of real numbers should be a complete ordered field. Thus we want to add completeness to the list of properties that the field of real numbers should possess.

Experience shows that no additional properties beyond being a complete ordered field are needed to prove all known properties of the real numbers. Thus we define the real numbers (which we have not previously defined) to be a complete ordered field.

Here is the formal definition, which is followed by a discussion of philosophical issues that this definition raises.

0.20 Definition $\mathbf{R}$, *the field of real numbers*

- The symbol $\mathbf{R}$ denotes a complete ordered field.
- $\mathbf{R}$ is called the field of *real numbers*.

The definition of $\mathbf{R}$ that we have just given raises the following three philosophical issues:

- **Definitions or axioms?** We have defined $\mathbf{R}$ to be a complete ordered field. For the rest of this book, we will prove results about $\mathbf{R}$ based on this definition. An alternative approach, in the spirit of the ancient Greeks, would be to list axioms (such as that every nonempty set with an upper bound has a least upper bound) that are assumed to be self-evident for the real line (which would not be defined in this approach).

  The more modern viewpoint taken here dispenses with axioms about $\mathbf{R}$ and instead concentrates on proving results about a certain structure—complete ordered fields—that is independent of any supposedly self-evident properties.

- **Existence?** We will be proving theorems about $\mathbf{R}$, which means theorems about complete ordered fields. Those theorems would be meaningless if there do not exist any complete ordered fields. Thus in the next section we will construct a complete ordered field.

- **Uniqueness?** The question of uniqueness is far less important than the question of existence. If there exist many different complete ordered fields, then all the theorems that we prove will apply to all those different complete ordered fields, which is a fine situation. However, you may be interested to know that there is essentially only one complete ordered field.

  Here *essentially only one* means that any two complete ordered fields differ only in the names of their elements. Specifically, Exercise 16 in Section C shows that there is a one-to-one function from any complete ordered field onto any other complete ordered field that preserves all the properties of a complete ordered field.

EXERCISES A

1 Prove 0.3.

2 State and prove properties for the multiplicative inverse in a field that are analogous to the properties in 0.3.

3 Suppose **F** is a field and $a, b \in \mathbf{F}$. Prove that $(-a)(-b) = ab$.

4 Suppose **F** is a field and $a, b, c \in \mathbf{F}$, with $b \neq 0$ and $c \neq 0$. Prove that

$$\frac{ac}{bc} = \frac{a}{b}.$$

5 Suppose **F** is a field and $a, b, c, d \in \mathbf{F}$, with $b \neq 0$ and $d \neq 0$. Prove that

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}.$$

6 Suppose **F** is a field and $a, b, c, d \in \mathbf{F}$, with $b \neq 0$, $c \neq 0$, and $d \neq 0$. Prove that

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}.$$

7 Suppose **F** is an ordered field and $a, b, c, d \in \mathbf{F}$. Prove that if $a < b$ and $c \leq d$, then $a + c < b + d$.

8 Suppose **F** is an ordered field and $a, b, c, d \in \mathbf{F}$. Prove that if $0 \leq a < b$ and $0 < c \leq d$, then $ac < bd$.

9 Suppose **F** is an ordered field and $a, b \in \mathbf{F}$. Prove that if $a < b$ and $ab > 0$, then $a^{-1} > b^{-1}$.

10 Prove that if a and b are elements of an ordered field, then $|ab| = |a||b|$.

11 Prove that if a and b are elements of an ordered field, then $\big||a| - |b|\big| \leq |a - b|$.

12 Prove that every ordered field has at most one positive element whose square equals 2 (where 2 is defined to be $1 + 1$).

13 Suppose **F** is an ordered field. Prove that there does not exist $i \in \mathbf{F}$ such that $i^2 = -1$. (Thus the set of complex numbers, with its usual operation of multiplication, cannot be made into an ordered field.)

14 Suppose **F** is the field of rational functions with coefficients in **R**. This means that an element of **F** has the form $\frac{p}{q}$, where p and q are polynomials with real coefficients and q is not the 0 polynomial. Rational functions $\frac{p}{q}$ and $\frac{r}{s}$ are declared to be equal if $ps = rq$, and addition and multiplication are defined in **F** as you would naturally assume.

(a) Let P denote the subset **F** consisting of rational functions that can be written in the form $\frac{p}{q}$, where the highest order terms of p and q both have positive coefficients. Show that **F** is an ordered field with this definition of P.

(b) Show that **F**, with P defined as above, is not a complete ordered field.

B Construction of the Real Numbers: Dedekind Cuts

You may intuitively think of a real number as a point on the real line (whatever that is) or as an integer followed by a decimal point followed by an infinite string of digits. Neither of those intuitive notions provides enough structure to easily verify the properties of a complete ordered field. Even defining the sum and the product of two real numbers can be difficult with those two approaches. Furthermore, these approaches make it hard to give a rigorous verification of the completeness property.

We will construct the real numbers using what are called Dedekind cuts. This clever construction uses only rational numbers, with no prior knowledge assumed about real numbers.

In 1858 Richard Dedekind (1831–1916) invented the simple but rigorous construction of the real numbers that have been named after him.

0.21 Definition ***Dedekind cut; $\mathcal{D}$***

A *Dedekind cut* is a nonempty subset D of $\mathbf{Q}$ with the following properties:

- $D \neq \mathbf{Q}$;
- D contains all rational numbers less than any element of D; in other words, if $b \in D$, then $a \in D$ for all $a \in \mathbf{Q}$ with $a < b$;
- D does not contain a largest element.

The set of all Dedekind cuts is denoted by $\mathcal{D}$.

The last bullet point above implies, for example, that $\{a \in \mathbf{Q} : a \leq 3\}$ is not a Dedekind cut because it contains a largest element (namely, 3). Note that all the following examples of Dedekind cuts are defined only in terms of rational numbers.

Intuitively, a Dedekind cut is the set of all rational numbers to the left of some point on the real line.

0.22 Example *Dedekind cuts*

Each of the following sets is a Dedekind cut:

- $\{a \in \mathbf{Q} : a < 3\}$;
- $\{a \in \mathbf{Q} : a < 0 \text{ or } a^2 < 2\}$;
- $\{a \in \mathbf{Q} : a < 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \text{ for some positive integer } n\}$.

The set in the first bullet point above consists of all rational numbers less than 3. If we had already defined the real numbers, then we could say that the set in the second bullet point consists of all rational numbers less than $\sqrt{2}$ and the set in the third bullet point consists of all rational numbers less than e. However, the Dedekind cuts defined above make sense even if we know nothing about the real numbers.

As we will see, the set $\mathcal{D}$ of all Dedekind cuts can be given the structure of a complete ordered field. Intuitively, we identify a Dedekind cut with the real number that would be its right endpoint on the real line if that made sense. In other words, think about the Dedekind cut in the first bullet point in the previous example as identified with 3, the Dedekind cut in the second bullet point as identified with $\sqrt{2}$, and the Dedekind cut in the third bullet point as identified with e.

Dedekind defined his cuts slightly differently than is done here. Specifically, Dedekind defined his cuts as ordered pairs of sets of rational numbers, with each number in the first set less than each number in the second set, and with the union of the two sets equaling $\mathbf{Q}$*. The approach taken here is a bit simpler, while still using Dedekind's basic idea.*

To give the set $\mathcal{D}$ of all Dedekind cuts the structure of a field, first we define addition on $\mathcal{D}$, along with the additive identity and additive inverses. You should think about why these are the right definitions.

0.23 Definition ***sum of two Dedekind cuts***

- Suppose C and D are Dedekind cuts. Then $C + D$ is defined by
$$C + D = \{c + d : c \in C, d \in D\}.$$
- The Dedekind cut $\widetilde{0}$ is defined by
$$\widetilde{0} = \{a \in \mathbf{Q} : a < 0\}.$$
- Suppose D is a Dedekind cut. Then $-D$ is defined by
$$-D = \{a \in \mathbf{Q} : a < -b \text{ for some } b \in \mathbf{Q} \text{ with } b \notin D\}.$$

The next result states that addition is well defined, that addition is commutative, that addition is associative, that $\widetilde{0}$ is the additive identity, and that $-D$ is the additive inverse of D for each Dedekind cut D. These properties help make $\mathcal{D}$ into a field.

0.24 ***addition of Dedekind cuts***

(a) $C + D$ is a Dedekind cut for all Dedekind cuts C, D.

(b) $C + D = D + C$ for all Dedekind cuts C, D.

(c) $(B + C) + D = B + (C + D)$ for all Dedekind cuts B, C, D.

(d) $D + \widetilde{0} = D$ for every Dedekind cut D.

(e) $-D$ is a Dedekind cut for every Dedekind cut D.

(f) $D + (-D) = \widetilde{0}$ for every Dedekind cut D.

The proof of the result above is left as an exercise.

The next step in making the set of all Dedekind cuts $\mathcal{D}$ into a field requires that we define multiplication. A definition analogous to the definition for the sum of two Dedekind cuts would define the product of two Dedekind cuts C and D to be the set of rational numbers less than or equal to cd for some $c \in C, d \in D$. However, that definition does not work because each Dedekind cut contains negative numbers with arbitrarily large absolute values. If we consider only Dedekind cuts C and D that each contain at least one positive number, then clearly we should define

Part (d) of the previous result states that $\widetilde{0}$ is the additive identity for the set $\mathcal{D}$ of all Dedekind cuts. Usually the additive identity in a field is called 0, but here we use the notation $\widetilde{0}$ to avoid confusion with the rational number 0.

$$CD = \{a \in \mathbf{Q} : a \leq cd \text{ for some } c \in C, d \in D \text{ with } c > 0, d > 0\},$$

with similarly appropriate definitions for the other three cases concerning C and D. For details, see the instruction before Exercise 4 in this section. That exercise ends with the conclusion that $\mathcal{D}$ (with the operations of addition and multiplication as defined) is a field.

Now that we have made $\mathcal{D}$ into a field, we want to make it into an ordered field. Thus we must define the positive subset of $\mathcal{D}$, which is done in the next definition. To motivate this definition, think of the intuitive notion of a Dedekind cut as corresponding to what should be its right endpoint.

0.25 Definition ***positive Dedekind cut***

A Dedekind cut D is called *positive* if $b > 0$ for some $b \in D$.

Exercise 5 asks you to verify that the definition above satisfies the requirements for the positive subset of a field (see 0.5). In other words, the definition above makes $\mathcal{D}$ into an ordered field.

Now that $\mathcal{D}$ is an ordered field, the positive subset of $\mathcal{D}$ defines the meaning of inequalities in the usual way (see 0.7). For example, $C \leq D$ means that $D + (-C)$ is positive or $C = D$. The next result shows that this ordering of $\mathcal{D}$ has a particularly nice interpretation. Again, the proof is left as an exercise.

0.26 ***ordering of Dedekind cuts***

Suppose C and D are Dedekind cuts. Then $C \leq D$ if and only if $C \subseteq D$.

Now we are ready to prove the main point about what we have been doing with Dedekind cuts. Specifically, we will prove that the ordered field $\mathcal{D}$ of Dedekind cuts is complete. The clean, easy proof of this result should be attributed to the cleverness of the definition of Dedekind cuts.

0.27 ***completeness of $\mathcal{D}$***

The ordered field $\mathcal{D}$ is complete.

Proof Suppose $\mathcal{A}$ is a nonempty subset of $\mathcal{D}$ that has an upper bound. We must show that $\mathcal{A}$ has a least upper bound.

Let

$$B = \{b \in \mathbf{Q} : b \in D \text{ for some } D \in \mathcal{A}\}.$$

In other words, B is the union of all the Dedekind cuts in $\mathcal{A}$.

We will show that B is a least upper bound of $\mathcal{A}$. For this to even make sense, we must first verify that B is a Dedekind cut. The following bullet points provide that verification:

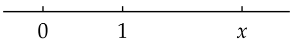
- Clearly B is nonempty, because $\mathcal{A}$ is nonempty.
- To show that $B \neq \mathbf{Q}$, we use the hypothesis that $\mathcal{A}$ has an upper bound. Because that upper bound is a Dedekind cut and thus is not all of $\mathbf{Q}$, there exists $a \in \mathbf{Q}$ not in that upper bound. Thus for every $D \in \mathcal{A}$, we see that $a \notin D$. Thus $a \notin B$. Hence $B \neq \mathbf{Q}$.
- The definition of B clearly implies that if $b \in B$, then $a \in B$ for all $a \in \mathbf{Q}$ with $a < b$.
- To show that B has no largest element, suppose $b \in B$. Then $b \in D$ for some $D \in \mathcal{A}$. Because D is a Dedekind cut, b is not a largest element of D. Because $D \subseteq B$, this implies that b is not a largest element of B. Thus B has no largest element, completing the verification that B is a Dedekind cut.

Now it makes sense to show that B is a least upper bound of $\mathcal{A}$. As usual, this least-upper-bound proof consists of two parts, first showing that B is an upper bound of $\mathcal{A}$ and then showing that B is less than or equal to all other upper bounds of $\mathcal{A}$:

- Obviously $D \subseteq B$ for every $D \in \mathcal{A}$, and thus $D \leq B$ for every $D \in \mathcal{A}$ (by 0.26). Hence B is an upper bound of $\mathcal{A}$.
- To show that B is a least upper bound of $\mathcal{A}$, suppose $C \in \mathcal{D}$ is an upper bound of $\mathcal{A}$. Thus $D \leq C$ for every $D \in \mathcal{A}$, which can be restated (by 0.26) as $D \subseteq C$ for every $D \in \mathcal{A}$. Because B is the union of all $D \in \mathcal{A}$, this implies that $B \subseteq C$. In other words, $B \leq C$. Hence B is a least upper bound of $\mathcal{A}$, completing the proof. ∎

What is a real number? The result above implies that we could think of $\mathbf{R}$ as $\mathcal{D}$, which would mean that a real number is a Dedekind cut. Although that viewpoint is technically correct, you may find it more useful to think of a real number intuitively as a point on the real line or as an element of an abstract complete ordered field.

Recall that $\mathbf{R}$ *is defined to be a complete ordered field. The existence of a complete ordered field, as proved in the previous result, means that the theorems proved in the rest of this book are not meaningless.*



A number x on the real line.

EXERCISES B

1 Prove 0.24 (the addition properties on the set of Dedekind cuts).

2 Prove that a Dedekind cut D is positive if and only if $0 \in D$.

3 Prove 0.26. In other words, show that if C and D are Dedekind cuts, then $C \leq D$ if and only if C is a subset of D.

For a Dedekind cut D, define the set difference $\mathbf{Q} \setminus D$ by

$$\mathbf{Q} \setminus D = \{r \in \mathbf{Q} : r \notin D\}$$

and define D^+ and D^- by

$$D^+ = \{d \in D : d > 0\} \quad \textit{and} \quad D^- = \{r \in \mathbf{Q} \setminus D : r \leq 0\}.$$

Think of the condition $D^+ \neq \emptyset$ as equivalent to $D > 0$. Now define the product CD of two Dedekind cuts C and D as follows:

$$CD = \begin{cases} \{cd : c \in C^+, d \in D^+\} \cup \{q \in \mathbf{Q} : q \leq 0\} & \textit{if } C^+ \neq \emptyset, D^+ \neq \emptyset; \\ \{cr : c \in C, r \in \mathbf{Q} \setminus D\} & \textit{if } C^+ = \emptyset, D^+ \neq \emptyset; \\ \{rd : r \in \mathbf{Q} \setminus C, d \in D\} & \textit{if } C^+ \neq \emptyset, D^+ = \emptyset; \\ \{a \in \mathbf{Q} : a < rs \textit{ for some } r \in C^-, s \in D^-\} & \textit{if } C^+ = \emptyset, D^+ = \emptyset. \end{cases}$$

You should think about why the definition above works to capture your intuitive expectations for multiplication of Dedekind cuts.

4 (a) Prove that if C and D are Dedekind cuts, then CD (as defined above) is a Dedekind cut.

(b) Let $\widetilde{1} = \{a \in \mathbf{Q} : a < 1\}$. Show that $\widetilde{1}$ is a multiplicative identity for the set $\mathcal{D}$ of all Dedekind cuts.

(c) For D a Dedekind cut with $D \neq \widetilde{0}$, find a formula for a Dedekind cut D^{-1} such that $DD^{-1} = \widetilde{1}$.

(d) Prove that with the addition and multiplication we have defined, the set $\mathcal{D}$ of all Dedekind cuts is a field.

5 Show that the definition of the positive subset of $\mathcal{D}$ as given in 0.25 satisfies the requirements for an ordered field (see 0.5). In other words, show the following:

(a) If D is a Dedekind cut, then D is positive or $D = \widetilde{0}$ or $-D$ is positive.

(b) If D is a positive Dedekind cut, then $-D$ is not a positive Dedekind cut.

(c) If C and D are positive Dedekind cuts, then $C + D$ and CD are positive.

6 Because $\mathcal{D}$ is an ordered field, the absolute value is defined on $\mathcal{D}$ (see 0.9). Prove that $|D| = D \cup (-D)$ for every Dedekind cut D.

C Supremum and Infimum

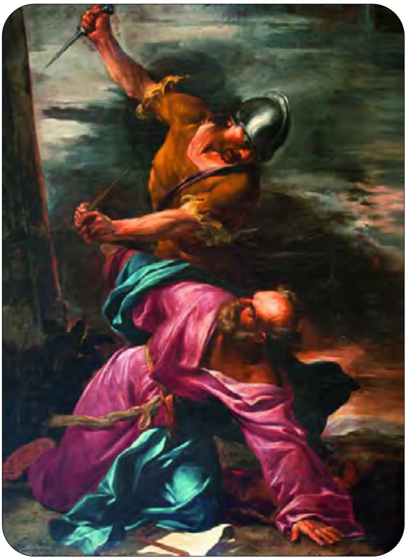
Archimedean Property

Let $\mathbf{Z}$ denote the set of integers and $\mathbf{Z}^+$ denote the set of positive integers. Thus

$$\mathbf{Z}^+ \subseteq \mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R},$$

where the last inclusion comes from the result 0.11.

The Archimedean Property states that if t is a real number, then there is a positive integer larger than t. This result surely does not surprise you. What may surprise you, however, is that this result cannot be proved using only the properties of an ordered field. Completeness must be used in every proof of the Archimedean Property because there are ordered fields in which this result fails (for an example, see Exercise 9 in this section).



The death of Archimedes, as depicted in a seventeenth-century painting.

0.28 *Archimedean Property*

Suppose $t \in \mathbf{R}$. Then there is a positive integer n such that $t < n$.

Proof Suppose there does not exist a positive integer n such that $t < n$. This implies that t is an upper bound of $\mathbf{Z}^+$. Because $\mathbf{R}$ is complete, this implies that $\mathbf{Z}^+$ has a least upper bound, which we will call b.

Now $b - 1$ is not an upper bound of $\mathbf{Z}^+$ (because b is the least upper bound of $\mathbf{Z}^+$). Thus there exists $m \in \mathbf{Z}^+$ such that $b - 1 < m$. Thus $b < m + 1$. Because $m + 1 \in \mathbf{Z}^+$, this contradicts the property that b is an upper bound of $\mathbf{Z}^+$. This contradiction completes the proof. ■

The next result gives a useful restatement of the Archimedean Property.

0.29 *Archimedean Property*

Suppose $\varepsilon \in \mathbf{R}$ and $\varepsilon > 0$. Then there is a positive integer n such that $\frac{1}{n} < \varepsilon$.

Proof In the first version of the Archimedean Property (0.28), let $t = \frac{1}{\varepsilon}$. ■

Now we have an important consequence of the Archimedean Property.

0.30 *rational number between every two distinct real numbers*

Suppose $a, b \in \mathbf{R}$, with $a < b$. Then there exists a rational number c such that $a < c < b$.

Proof First suppose $a \geq 0$. By the Archimedean Property (0.29), there is a positive integer n such that

$$\frac{1}{n} < b - a.$$

Let

$$\mathcal{A} = \Big\{m \in \mathbf{Z} : a < \frac{m}{n}\Big\}.$$

By the Archimedean Property (0.28), there is a positive integer m such that $an < m$. Thus $\mathcal{A}$ is a nonempty set of positive integers. Hence $\mathcal{A}$ has a smallest element, which we will call M. Because $M \in \mathcal{A}$, we have $a < \frac{M}{n}$.

Now $M - 1 \notin \mathcal{A}$ (because M is the smallest element of $\mathcal{A}$). Thus $\frac{M-1}{n} \leq a$, which implies that

$$\begin{aligned}\frac{M}{n} &\leq a + \frac{1}{n}\\ &< b.\end{aligned}$$

Hence taking $c = \frac{M}{n}$ completes the proof in the case where $a \geq 0$.

Now suppose $a < 0$. If $b > 0$, then take $c = 0$. If $b \leq 0$, then apply the previous case to find a rational number d such that $-b < d < -a$, then take $c = -d$. ■

Greatest Lower Bound

The concepts of upper bound and least upper bound played a key role in our development of the notion of a complete ordered field. Now we make the situation more symmetric by introducing the concepts of lower bound and greatest lower bound.

0.31 Definition *lower bound*

Suppose $A \subseteq \mathbf{R}$. A number $b \in \mathbf{R}$ is called a *lower bound* of A if $b \leq a$ for every $a \in A$.

0.32 Definition *greatest lower bound*

Suppose $A \subseteq \mathbf{R}$. A number $b \in \mathbf{R}$ is called a *greatest lower bound* of A if both the following conditions hold:

- b is a lower bound of A;
- $b \geq c$ for every lower bound c of A.

If a subset of $\mathbf{R}$ has a greatest lower bound, then the subset has a unique greatest lower bound. The uniqueness follows from the same reasoning as for the uniqueness of the least upper bound (see the comment before 0.17).

0.33 **Example** *greatest lower bounds*

If

$$A_1 = \{a \in \mathbf{R} : 3 < a < 5\} \quad \text{and} \quad A_2 = \{a \in \mathbf{R} : 3 \leq a \leq 5\},$$

then every real number b with $b \leq 3$ is a lower bound of A_1 and of A_2. Thus 3 is the greatest lower bound of both A_1 and A_2.

The completeness property of the real numbers tells us that every nonempty subset of $\mathbf{R}$ with an upper bound has a least upper bound. The result below is the corresponding statement for lower bounds. Note that the upper bound property is part of the definition of $\mathbf{R}$, but the lower bound property below is a theorem.

0.34 ***existence of greatest lower bound***

Every nonempty subset of $\mathbf{R}$ that has a lower bound has a greatest lower bound.

Proof Suppose A is a nonempty subset of $\mathbf{R}$ that has a lower bound b. Let

$$-A = \{-a : a \in A\}.$$

Then $-b$ is an upper bound of $-A$, as you should verify. The completeness of $\mathbf{R}$ implies that $-A$ has a least upper bound, which we will call t. Now $-t$ is a greatest lower bound of A, as you should verify. ∎

The terminology defined below has wide usage in many areas of mathematics.

0.35 **Definition** ***supremum and infimum***

Suppose $A \subseteq \mathbf{R}$. The *supremum* of A, denoted $\sup A$, is defined as follows:

$$\sup A = \begin{cases} \text{the least upper bound of } A & \text{if } A \text{ has an upper bound and } A \neq \varnothing, \\ \infty & \text{if } A \text{ does not have an upper bound,} \\ -\infty & \text{if } A = \varnothing. \end{cases}$$

The *infimum* of A, denoted $\inf A$ is defined as follows:

$$\inf A = \begin{cases} \text{the greatest lower bound of } A & \text{if } A \text{ has a lower bound and } A \neq \varnothing, \\ -\infty & \text{if } A \text{ does not have a lower bound,} \\ \infty & \text{if } A = \varnothing. \end{cases}$$

The term *supremum*, which comes from the same Latin root as the word *superior*, should help remind you that $\sup A$ is trying to be the largest number in A (if $\sup A \in A$, then $\sup A$ is the largest number in A). Similarly, the term *infimum*, which comes from the same Latin root as the word *inferior*, should help remind you that $\inf A$ is trying to be the smallest number in A (if $\inf A \in A$, then $\inf A$ is the smallest number in A).

0.36 **Example** *infimum and supremum*

- If $A_1 = \{a \in \mathbf{R} : 3 < a < 5\}$ and $A_2 = \{a \in \mathbf{R} : 3 \le a \le 5\}$, then
$$\inf A_1 = \inf A_2 = 3 \quad \text{and} \quad \sup A_1 = \sup A_2 = 5.$$
- If $A = \{1 - \frac{1}{n} : n \in \mathbf{Z}^+\} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$, then $\inf A = 0$ and $\sup A = 1$.

The symbols ∞ and $-\infty$ that appear in the definitions of supremum and infimum do not represent real numbers. The equation $\sup A = \infty$ is simply an abbreviation for the statement A *does not have an upper bound*. Similarly, the equation $\inf A = -\infty$ is an abbreviation for the statement A *does not have a lower bound*.

Irrational Numbers

The completeness property of the real numbers implies the existence of a real number whose square is 2.

0.37 ***existence of*** $\sqrt{2}$

There is a positive real number whose square equals 2.

Proof Let
$$b = \sup\{a \in \mathbf{R} : a^2 < 2\}.$$
The set $\{a \in \mathbf{R} : a^2 < 2\}$ has an upper bound (for example, 2 is an upper bound) and thus b as defined above is a real number.

If $b^2 < 2$, then we can find a number slightly bigger than b in $\{a \in \mathbf{R} : a^2 < 2\}$ (see the second paragraph of Example 0.18 for this calculation), which contradicts the property that b is an upper bound of $\{a \in \mathbf{R} : a^2 < 2\}$.

If $b^2 > 2$, then we can find a number slightly smaller than b that is an upper bound of $\{a \in \mathbf{R} : a^2 < 2\}$ (see the third paragraph of Example 0.18 for this calculation), which contradicts the property that b is the least upper bound of $\{a \in \mathbf{R} : a^2 < 2\}$.

The two previous paragraphs imply that $b^2 = 2$, as desired. ∎

The real number b produced by the proof above is called the *square root* of 2 and is denoted by $\sqrt{2}$.

0.38 **Definition** ***irrational number***

A real number is called *irrational* if it is not rational.

For example, 0.12 shows that $\sqrt{2}$ is irrational. Other well-known irrational numbers include e, π, and $\ln 2$.

We showed previously that there is a rational number between every two distinct real numbers (see 0.30). Now we can show that there is also an irrational number between every two distinct real numbers.

0.39 ***irrational number between every two distinct real numbers***

Suppose $a, b \in \mathbf{R}$, with $a < b$. Then there exists an irrational number c such that $a < c < b$.

Proof By the Archimedean Property (0.29), there is a positive integer n such that $\frac{1}{n} < b - a$. Let

$$c = \begin{cases} a + \frac{\sqrt{2}}{2n} & \text{if } a \text{ is rational,} \\ a + \frac{1}{n} & \text{if } a \text{ is irrational.} \end{cases}$$

Then c is irrational and $a < c < b$. ■

Are the numbers

$$e + \pi, \quad e\pi, \quad \frac{\pi}{e}$$

rational or irrational? No one has been able to answer this question. However, almost all mathematicians suspect that all three of these numbers are irrational.

Intervals

We will find it useful sometimes to consider a set (not a field) consisting of $\mathbf{R}$ and two additional elements called ∞ and $-\infty$. We define an ordering on $\mathbf{R} \cup \{\infty, -\infty\}$ to behave exactly as you expect from the names of the two additional symbols.

0.40 Definition ***ordering on*** $\mathbf{R} \cup \{\infty, -\infty\}$

- The ordering $<$ on $\mathbf{R}$ is extended to $\mathbf{R} \cup \{\infty, -\infty\}$ as follows:

  $a < \infty$ for all $a \in \mathbf{R} \cup \{-\infty\}$;

  $-\infty < a$ for all $a \in \mathbf{R} \cup \{\infty\}$.

- For $a, b \in \mathbf{R} \cup \{\infty, -\infty\}$,

  the notation $a \leq b$ means that $a < b$ or $a = b$;

  the notation $a > b$ means that $b < a$;

  the notation $a \geq b$ means that $a > b$ or $a = b$.

The notation defined below is probably already familiar to you.

0.41 Definition ***interval notation***

Suppose $a, b \in \mathbf{R} \cup \{\infty, -\infty\}$. Then

- $(a, b) = \{t \in \mathbf{R} : a < t < b\}$;
- $[a, b] = \{t \in \mathbf{R} \cup \{\infty, -\infty\} : a \leq t \leq b\}$;
- $(a, b] = \{t \in \mathbf{R} \cup \{\infty\} : a < t \leq b\}$;
- $[a, b) = \{t \in \mathbf{R} \cup \{-\infty\} : a \leq t < b\}$.

If $a > b$, then all four of the sets listed above are the empty set. If $a = b$, then $[a, b]$ is the set $\{a\}$ containing only one element and the other three sets listed above are the empty set.

The definition above implies that $(-\infty, \infty)$ equals $\mathbf{R}$ and that $(0, \infty)$ is the set of positive numbers. Also note that $[-\infty, \infty] = \mathbf{R} \cup \{\infty, -\infty\}$ and that $[0, \infty] = [0, \infty) \cup \{\infty\}$; thus neither $[-\infty, \infty]$ nor $[0, \infty]$ is a subset of $\mathbf{R}$.

0.42 Definition ***interval***

- A subset of $[-\infty, \infty]$ is called an *interval* if it contains all numbers that are between pairs of its elements.
- In other words, a set $I \subseteq [-\infty, \infty]$ is called an *interval* if $c, d \in I$ implies $(c, d) \subseteq I$.

The next result gives a complete description of all intervals of $[-\infty, \infty]$.

0.43 ***description of intervals***

Suppose $I \subseteq [-\infty, \infty]$ is an interval. Then I is one of the following sets for some $a, b \in [-\infty, \infty]$:

$$(a, b), \quad [a, b], \quad (a, b], \quad [a, b).$$

Proof Let $a = \inf I$ and let $b = \sup I$. Suppose $s \in I$. Then $a \leq s$ because a is a lower bound of I. Similarly, $s \leq b$. Thus $s \in [a, b]$. We have shown that $I \subseteq [a, b]$.

Now suppose $t \in (a, b)$. Because $a < t$ and a is the greatest lower bound of I, the number t is not a lower bound of I. Thus there exists $c \in I$ such that $c < t$. Similarly, because $t < b$, there exists $d \in I$ such that $t < d$. Hence $t \in (c, d)$. Because $c, d \in I$ and I is an interval, we can conclude that $t \in I$. Hence we have shown that $(a, b) \subseteq I$.

We now know that

$$(a, b) \subseteq I \subseteq [a, b].$$

This implies that I is (a, b), $[a, b]$, $(a, b]$ or $[a, b)$. ■

EXERCISES C

1 Suppose $b \in \mathbf{R}$ and $|b| < \frac{1}{n}$ for every positive integer n. Prove that $b = 0$.

2 Suppose $A \subseteq B \subseteq \mathbf{R}$. Show that

$$\inf A \geq \inf B \quad \text{and} \quad \sup A \leq \sup B.$$

3 Explain why it makes no sense to inquire about whether your current height as measured in meters is a rational number or an irrational number.

4 Is the speed of light in a vacuum, as measured in meters per second, a rational number or an irrational number?

5 Prove or give a counterexample: Suppose $a_1, a_2, a_3, \ldots$ is a sequence of rational numbers such that

$$\sup\{a_1, a_2, a_3, \ldots\} = \sqrt{2}.$$

Then

$$\sup\{a_n, a_{n+1}, a_{n+2}, \ldots\} = \sqrt{2}$$

for every positive integer n.

For A and B nonempty subsets of $\mathbf{R}$, define $A + B$ by

$$\boldsymbol{A + B = \{a + b : a \in A, b \in B\}.}$$

Define arithmetic with ∞ and $-\infty$ as you would expect. For example, $s + \infty = \infty$ for all $s \in (-\infty, \infty]$ and $-\infty + t = -\infty$ for all $t \in [-\infty, \infty)$. Note, however, that $\infty + (-\infty)$ should remain undefined.

6 Prove that if A and B are nonempty subsets of $\mathbf{R}$, then

$$\sup(A + B) = \sup A + \sup B$$

and

$$\inf(A + B) = \inf A + \inf B.$$

An expression such as $\sup\limits_X f$ means $\sup\{f(x) : x \in X\}$.
Similarly, $\inf\limits_X f(x)$ means $\inf\{f(x) : x \in X\}$.

7 Suppose X is a nonempty set and $f, g \colon X \to \mathbf{R}$ are functions.

(a) Prove that

$$\sup_X (f + g) \leq \sup_X f + \sup_X g.$$

(b) Give an example to show that the inequality above can be a strict inequality.

8 Suppose X is a nonempty set and $f, g \colon X \to \mathbf{R}$ are functions.

(a) Prove that

$$\inf_X (f + g) \geq \inf_X f + \inf_X g.$$

(b) Give an example to show that the inequality above can be a strict inequality.

9 Prove that the ordered field of rational functions with coefficients in $\mathbf{R}$ (see Exercise 14 in Section A for the definition of this ordered field) does not satisfy the Archimedean Property.

10 (a) Suppose $a_1, a_2, \ldots$ is a sequence in $\mathbf{R}$. Prove that

$$\inf\{a_m, a_{m+1}, \ldots\} \leq \sup\{a_n, a_{n+1}, \ldots\}$$

for all positive integers m, n.

(b) Describe all sequences $a_1, a_2, \ldots$ in $\mathbf{R}$ such that the inequality above is an equality for some positive integers m, n.

11 Suppose $I \subseteq [-\infty, \infty]$ is an interval. Prove that the number of elements in $I \cap \mathbf{Q}$ is 0 or 1 or is not finite.

12 Suppose A is a subset of $\mathbf{Q}$ that contains all the rational numbers that are between pairs of its elements. Prove that there exists an interval $I \subseteq \mathbf{R}$ such that $A = I \cap \mathbf{Q}$.

13 Prove or give a counterexample: The intersection of every collection of intervals is an interval.

14 Prove or give a counterexample: The union of every collection of intervals with a nonempty intersection is an interval.

15 Suppose $I \subseteq R$ is an interval containing more than one number. Prove that

$$\inf(I \cap \mathbf{Q}) = \inf I \quad \text{and} \quad \sup(I \cap \mathbf{Q}) = \sup I.$$

16 Suppose $\mathbf{R}_1$ and $\mathbf{R}_2$ are complete ordered fields. Let $\varphi_1 \colon \mathbf{Q} \to \mathbf{R}_1$ and $\varphi_2 \colon \mathbf{Q} \to \mathbf{R}_2$ be the functions that allow us to think of $\mathbf{Q}$ as a subset of $\mathbf{R}_1$ and as a subset of $\mathbf{R}_2$ (see 0.11). Define $\psi \colon \mathbf{R}_1 \to \mathbf{R}_2$ as follows: for $a \in \mathbf{R}_1$, let $\psi(a)$ be the least upper bound in $\mathbf{R}_2$ of

$$\{\varphi_2(q) : q \in \mathbf{Q} \text{ and } \varphi_1(q) \leq a\}.$$

(a) Show that ψ is a well-defined, one-to-one function from $\mathbf{R}_1$ onto $\mathbf{R}_2$.

(b) Show that $\psi(0) = 0$ and $\psi(1) = 1$.

(c) Show that $\psi(a + b) = \psi(a) + \psi(b)$ for all $a, b \in \mathbf{R}_1$.

(d) Show that $\psi(-a) = -\psi(a)$ for all $a \in \mathbf{R}_1$.

(e) Show that $\psi(ab) = \psi(a)\psi(b)$ for all $a, b \in \mathbf{R}_1$.

(f) Show that $\psi(a^{-1}) = \big(\psi(a)\big)^{-1}$ for all $a \in \mathbf{R}_1$ with $a \neq 0$.

(g) Suppose $a \in \mathbf{R}_1$. Show that $a > 0$ if and only if $\psi(a) > 0$.

[*Items (a)–(g) above show that* $\mathbf{R}_1$ *and* $\mathbf{R}_2$ *are essentially the same as complete ordered fields, with* ψ *providing the relabeling.*]

D Open and Closed Subsets of $\mathbf{R}^n$

Limits in $\mathbf{R}^n$

Throughout the next two sections, assume that m and n are positive integers. Thus, for example, 0.48 should include the hypothesis that n is a positive integer, but theorems and definitions become easier to state without explicitly repeating this hypothesis.

We identify $\mathbf{R}^1$ with $\mathbf{R}$, the real line. The set $\mathbf{R}^2$, which you can think of as a plane, is the set of all ordered pairs of real numbers. The set $\mathbf{R}^3$, which you can think of as ordinary space, is the set of all ordered triples of real numbers. The next definition gives the obvious generalization to higher dimensions.

0.44 Definition $\mathbf{R}^n$

$\mathbf{R}^n$ is the set of all ordered n-tuples of real numbers:

$$\mathbf{R}^n = \{(x_1, \dots, x_n) : x_1, \dots, x_n \in \mathbf{R}\}.$$

Now we generalize to the setting of $\mathbf{R}^n$ the standard Euclidean distance from a point in $\mathbf{R}^2$ or $\mathbf{R}^3$ to the origin. We also introduce another measurement of distance that is a bit easier to manipulate.

0.45 Definition $\|\cdot\|$; $\|\cdot\|_\infty$

For $(x_1, \dots, x_n) \in \mathbf{R}^n$, let

$$\|(x_1, \dots, x_n)\| = \sqrt{{x_1}^2 + \cdots + {x_n}^2}$$

and

$$\|(x_1, \dots, x_n)\|_\infty = \max\{|x_1|, \dots, |x_n|\}.$$

The reason for using the subscript ∞ here will become clear when you get to L^p-spaces in Chapter 7 of *Measure, Integration & Real Analysis*. For now, note that the triangle inequality

$$\|x + y\|_\infty \leq \|x\|_\infty + \|y\|_\infty \text{ for all } x, y \in \mathbf{R}^n$$

is an easy consequence of the definition of $\|\cdot\|_\infty$. The triangle inequality also holds for $\|\cdot\|$ but its proof when $n \geq 3$ is far from obvious; *Measure, Integration & Real Analysis* contains two nice proofs of the triangle inequality for $\|\cdot\|$ (see 7.14 with $p = 2$ and 8.15). Meanwhile, here we will use $\|\cdot\|_\infty$ for simpler proofs. Note that if $n = 1$, then $\|\cdot\|$ and $\|\cdot\|_\infty$ both equal the absolute value $|\cdot|$.

Now we are ready to define what it means for a sequence of elements of $\mathbf{R}^n$ to have a limit. The intuition concerning limits is that if we go far enough out in a sequence, then all the terms beyond that will be as close as we wish to the limit. You should have seen limits in previous courses. Thus some key properties of limits are left as exercises.

0.46 Definition ***limit***

Suppose $a_1, a_2, \ldots \in \mathbf{R}^n$ and $L \in \mathbf{R}^n$. Then L is called a *limit* of the sequence $a_1, a_2, \ldots$ and we write

$$\lim_{k\to\infty} a_k = L$$

if for every $\varepsilon > 0$, there exists $m \in \mathbf{Z}^+$ such that

$$\|a_k - L\|_\infty < \varepsilon$$

for all integers $k \geq m$.

The definition above implies that

$$\lim_{k\to\infty} a_k = L \text{ if and only if } \lim_{k\to\infty} \|a_k - L\|_\infty = 0.$$

Because

$$\|x\|_\infty \leq \|x\| \leq \sqrt{n}\|x\|_\infty \text{ for all } x \in \mathbf{R}^n,$$

we see that

$$\lim_{k\to\infty} a_k = L \text{ if and only if } \lim_{k\to\infty} \|a_k - L\| = 0.$$

We will need the following useful terminology.

0.47 Definition ***converge; convergent***

A sequence in $\mathbf{R}^n$ is said to *converge* and to be a *convergent sequence* if it has a limit.

The next result states that a sequence of elements of $\mathbf{R}^n$ converges if and only if it converges coordinatewise. Thus questions about convergence of sequences in $\mathbf{R}^n$ can often be reduced to questions about convergence of sequences in $\mathbf{R}$. The proof of this next result is left to the reader.

0.48 ***coordinatewise limits***

Suppose $a_1, a_2, \ldots \in \mathbf{R}^n$ and $L \in \mathbf{R}^n$. For $k \in \mathbf{Z}^+$, let

$$(a_{k,1}, \ldots, a_{k,n}) = a_k,$$

and let $(L_1, \ldots, L_n) = L$. Then $\lim\limits_{k\to\infty} a_k = L$ if and only if

$$\lim_{k\to\infty} a_{k,j} = L_j$$

for each $j \in \{1, \ldots, n\}$.

You should show that each sequence in $\mathbf{R}^n$ has at most one limit. Thus the phrase *a limit* in 0.46 can be replaced by *the limit*.

Open Subsets of $\mathbf{R}^n$

If $n = 3$, then the open cube $B(x, \delta)$ defined below is the usual cube in $\mathbf{R}^3$ centered at x with sides of length 2δ.

0.49 Definition ***open cube***

For $x \in \mathbf{R}^n$ and $\delta > 0$, the *open cube* $B(x, \delta)$ is defined by

$$B(x, \delta) = \{y \in \mathbf{R}^n : \|y - x\|_\infty < \delta\}.$$

As a test that you are comfortable with these concepts, be sure that you can verify the following implication:

0.50 $$y \in B(x, \delta) \implies B(y, \delta - \|y - x\|_\infty) \subseteq B(x, \delta).$$

In the next definition, we allow the endpoints of an open interval to be $\pm\infty$.

0.51 Definition ***open interval***

A subset of $\mathbf{R}$ of the form (a, b) for some $a, b \in [-\infty, \infty]$ is called an *open interval*.

Note that if $n = 1$ and $x \in \mathbf{R}$, then $B(x, \delta)$ is the open interval $(x - \delta, x + \delta)$.

Two equivalent definitions of open subsets of $\mathbf{R}^n$ are given below. The definition in the first bullet point is more useful when generalizing to metric spaces. The definition in the second bullet point is more useful when considering bases of topological spaces.

0.52 Definition ***open subset of*** $\mathbf{R}^n$

- A subset G of $\mathbf{R}^n$ is called *open* if for every $x \in G$, there exists $\delta > 0$ such that $B(x, \delta) \subseteq G$.
- Equivalently, a subset G of $\mathbf{R}^n$ is called *open* if every element of G is contained in an open cube that is contained in G.

Make sure you take the time to understand why the definitions given by the two bullet points above are equivalent (you will need to use 0.50).

Open sets could have been defined using the open balls $\{y \in \mathbf{R}^n : \|y - x\| < \delta\}$ instead of the open cubes $B(x, \delta)$. These two possible approaches are equivalent because every open cube contains an open ball with the same center, and every open ball contains an open cube with the same center. Specifically, if $x \in \mathbf{R}^n$ and $\delta > 0$ then

$$\{y \in \mathbf{R}^n : \|y - x\| < \delta\} \subseteq B(x, \delta) \subseteq \{y \in \mathbf{R}^n : \|y - x\| < \sqrt{n}\delta\},$$

as you should verify.

The *union* $\bigcup_{k=1}^{\infty} E_k$ of a sequence $E_1, E_2, \ldots$ of subsets of a set S is the set of elements of S that are in at least one of the E_k. The *intersection* $\bigcap_{k=1}^{\infty} E_k$ is the set of elements of S that are in all the E_k.

More generally, we can consider unions and intersections that are not indexed by the positive integers.

0.53 Definition ***union and intersection***

Suppose $\mathcal{A}$ is a collection of subsets of some set S.

- The *union* of the collection $\mathcal{A}$, denoted $\bigcup_{E \in \mathcal{A}} E$, is defined by

$$\bigcup_{E \in \mathcal{A}} E = \{x \in S : x \in E \text{ for some } E \in \mathcal{A}\}.$$

- The *intersection* of the collection $\mathcal{A}$, denoted $\bigcap_{E \in \mathcal{A}} E$, is defined by

$$\bigcap_{E \in \mathcal{A}} E = \{x \in S : x \in E \text{ for every } E \in \mathcal{A}\}.$$

0.54 **Example** *union and intersection*

$$\bigcup_{k=1}^{\infty} \left[\tfrac{1}{k}, 1 - \tfrac{1}{k}\right] = (0, 1) \quad \text{and} \quad \bigcap_{k=1}^{\infty} \left(-\tfrac{1}{k}, \tfrac{1}{k}\right) = \{0\}$$

0.55 ***union and intersection of open sets***

(a) The union of every collection of open subsets of $\mathbf{R}^n$ is an open subset of $\mathbf{R}^n$.

(b) The intersection of every finite collection of open subsets of $\mathbf{R}^n$ is an open subset of $\mathbf{R}^n$.

Proof The proof of (a) is left to the reader.

To prove (b), suppose $G_1, \ldots, G_m$ are open subsets of $\mathbf{R}^n$ and $x \in G_1 \cap \cdots \cap G_m$. Thus $x \in G_j$ for each $j = 1, \ldots, m$. Because each G_j is open, there exist positive numbers $\delta_1, \ldots, \delta_m$ such that $B(x, \delta_j) \subseteq G_j$ for each $j = 1, \ldots, m$. Let

$$\delta = \min\{\delta_1, \ldots, \delta_m\}.$$

Then $\delta > 0$ and $B(x, \delta) \subseteq G_1 \cap \cdots \cap G_m$. Thus $G_1 \cap \cdots \cap G_m$ is an open subset of $\mathbf{R}^n$, completing the proof. ∎

The conclusion of 0.55(b) cannot be strengthened to infinite intersections because 0.54 gives an example of a sequence of open subsets of $\mathbf{R}$ whose intersection is the set $\{0\}$, which is not open.

The next definition helps us distinguish some sets as having the same number of elements as $\mathbf{Z}^+$.

0.56 Definition ***countable; uncountable***

- A set C is called *countable* if $C = \varnothing$ or if $C = \{c_1, c_2, \dots\}$ for some sequence $c_1, c_2, \dots$ of elements of C.
- A set is called *uncountable* if it is not countable.

The following two points follow easily from the definition of a countable set.

- Every finite set is countable. This holds because if $C = \{c_1, c_2, \dots, c_n\}$, then $C = \{c_1, c_2, \dots, c_n, c_n, c_n, \dots\}$ (repetitions do not matter for sets).
- If C is an infinite countable set, then C can be written in the form $\{b_1, b_2, \dots\}$ where $b_1, b_2, \dots$ are all distinct. This holds because we can delete any terms in the sequence $c_1, c_2, \dots$ that appear earlier in the sequence.

We will use the next result to prove our description of open subsets of $\mathbf{R}$ (0.59).

0.57 ***$\mathbf{Q}$ is countable***

The set of rational numbers is countable.

Proof At step 1, start with the list $-1, 0, 1$. At step n, adjoin to the list in increasing order the rational numbers in the interval $[-n, n]$ that can be written in the form $\frac{m}{n}$ for some integer m. Thus halfway through step 3, the list is as follows:

$$-1, 0, 1, -2, -\tfrac{3}{2}, -1, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, -3, -\tfrac{8}{3}, -\tfrac{7}{3}, -2, -\tfrac{5}{3}, -\tfrac{4}{3}, -1, -\tfrac{2}{3}, -\tfrac{1}{3}, 0.$$

Continue in this fashion to produce a sequence that contains each rational number, completing the proof.

Deleting the entries in the list that already appear earlier in the list (shown above in red) produces a sequence that contains each rational number exactly once. ■

As you probably know, $\mathbf{R}$ is uncountable (see 2.17 in *Measure, Integration & Real Analysis* for a proof that may be new to you). Similarly, the set of irrational numbers is uncountable. Thus there are more irrational numbers than rational numbers.

The following terminology will be useful.

0.58 Definition ***disjoint***

A sequence $E_1, E_2, \dots$ of sets is called *disjoint* if $E_j \cap E_k = \varnothing$ whenever $j \neq k$.

The next result gives a complete description of the open subsets of $\mathbf{R}$. As an example of this result, the open set consisting of all real numbers that are not integers equals the union of the disjoint sequence of open intervals

$$(0, 1), (-1, 0), (1, 2), (-2, -1), (2, 3), (-3, -2), \dots.$$

In the result below, some (perhaps infinitely many) of the open intervals may be the empty set.

0.59 ***open subset of* R *is countable disjoint union of open intervals***

A subset of $\mathbf{R}$ is open if and only if it is the union of a disjoint sequence of open intervals.

Proof One direction of this result is easy: The union of every sequence (disjoint or not) of open intervals is open [by 0.55(a)].

To prove the other direction, suppose G is an open subset of $\mathbf{R}$. For each $t \in G$, let G_t be the union of all the open intervals contained in G that contain t. A moment's thought shows that G_t is the largest open interval contained in G that contains t.

If $s, t \in G$ and $G_s \cap G_t \neq \varnothing$, then $G_s = G_t$ (because otherwise $G_s \cup G_t$ would be an open interval strictly larger than at least one of G_s and G_t and containing both s and t). In other words, any two intervals in the collection of intervals $\{G_t : t \in G\}$ are either disjoint or equal to each other.

Because $t \in G_t$ for each $t \in G$, we see that the union of the collection of intervals $\{G_t : t \in G\}$ is G.

Let $r_1, r_2, \ldots$ be a sequence of rational numbers that includes every rational number (such a sequence exists by 0.57). Define a sequence of open intervals $I_1, I_2, \ldots$ as follows:

$$I_k = \begin{cases} \varnothing & \text{if } r_k \notin G, \\ \varnothing & \text{if } r_k \in I_j \text{ for some } j < k, \\ G_{r_k} & \text{if } r_k \in G \text{ and } r_k \notin I_j \text{ for all } j < k. \end{cases}$$

If $t \in G$, then the open interval G_t contains a rational number (by 0.30) and thus $G_t = I_k$ for some positive integer k. Thus G is the union of the disjoint sequence of open intervals $I_1, I_2, \ldots$. ∎

Closed Subsets of $\mathbf{R}^n$

0.60 Definition ***set difference; complement***

- If S and A are sets, then the *set difference* $S \setminus A$ is defined to be the set of elements of S that are not in A. In other words, $S \setminus A = \{s \in S : s \notin A\}$.
- If $A \subseteq S$, then $S \setminus A$ is called the *complement* of A in S.

For example, the complement in $\mathbf{R}$ of the interval $(-\infty, 5)$ is $[5, \infty)$.

0.61 Definition ***closed subset of*** $\mathbf{R}^n$

A subset of $\mathbf{R}^n$ is called *closed* if its complement in $\mathbf{R}^n$ is open.

For example, the interval $[1,4]$ is a closed subset of $\mathbf{R}$ because its complement in $\mathbf{R}$ is the open set $(-\infty,1)\cup(4,\infty)$.

Unlike doors, a subset of $\mathbf{R}^n$ need not be either open or closed. For example, the interval $(3,7]$ is neither an open nor a closed subset of $\mathbf{R}$.

Closed sets can be more complicated than open sets. There exist closed subsets of $\mathbf{R}$ that are not the union of a sequence of intervals (an example follows from 2.76 and 2.80 of *Measure, Integration & Real Analysis*).

The following characterization of closed sets will frequently be useful.

0.62 ***characterization of closed sets***

A subset of $\mathbf{R}^n$ is closed if and only if it contains the limit of every convergent sequence of elements of the set.

Proof We will prove the contrapositive in both directions.

First suppose A is a subset of $\mathbf{R}^n$ such that some convergent sequence $a_1, a_2, \dots$ of elements of A has a limit L that is not in A. Because $L = \lim_{k\to\infty} a_k$, for each $\delta > 0$ there exists $k \in \mathbf{Z}^+$ such that $\|L - a_k\|_\infty < \delta$. Thus $L \in \mathbf{R}^n \setminus A$ and

$$B(L,\delta) \not\subset \mathbf{R}^n \setminus A$$

for every $\delta > 0$. Hence $\mathbf{R}^n \setminus A$ is not an open subset of $\mathbf{R}^n$. Thus A is not a closed subset of $\mathbf{R}^n$, completing the proof in one direction.

To prove the other direction, now suppose A is a subset of $\mathbf{R}^n$ that is not closed. Thus $\mathbf{R}^n \setminus A$ is not open. Hence there exists $L \in \mathbf{R}^n \setminus A$ such that

$$B(L,\tfrac{1}{k}) \not\subset \mathbf{R}^n \setminus A$$

for every $k \in \mathbf{Z}^+$. Thus for each $k \in \mathbf{Z}^+$, there exists $a_k \in A$ such that

$$\|L - a_k\|_\infty < \tfrac{1}{k}.$$

The inequality above implies that the sequence $a_1, a_2, \dots$ of elements of A has limit L. Thus there exists a convergent sequence of elements of A whose limit is not in A, completing the proof in the other direction. ■

The next result can be stated without symbols: The complement of a union is the intersection of the complements, and the complement of an intersection is the union of the complements.

Augustus De Morgan (1806–1871) became the first professor of mathematics at University College London when he was 22 years old.

0.63 ***De Morgan's Laws***

Suppose $\mathcal{A}$ is a collection of subsets of some set X. Then

$$X \setminus \bigcup_{E\in\mathcal{A}} E = \bigcap_{E\in\mathcal{A}} (X\setminus E) \quad\text{and}\quad X \setminus \bigcap_{E\in\mathcal{A}} E = \bigcup_{E\in\mathcal{A}} (X\setminus E).$$

Proof An element $x \in X$ is not in $\bigcup_{E \in \mathcal{A}} E$ if and only if x is not in E for every $E \in \mathcal{A}$. Thus the first equality above holds.

An element $x \in X$ is not in $\bigcap_{E \in \mathcal{A}} E$ if and only if x is not in E for some $E \in \mathcal{A}$. Thus the second equality above holds. ∎

0.64 *union and intersection of closed sets*

(a) The intersection of every collection of closed subsets of $\mathbf{R}^n$ is a closed subset of $\mathbf{R}^n$.

(b) The union of every finite collection of closed subsets of $\mathbf{R}^n$ is a closed subset of $\mathbf{R}^n$.

Proof This result follows immediately from De Morgan's Laws (0.63), the definition of a closed set, and 0.55. ∎

The conclusion in 0.64(b) cannot be strengthened to infinite unions because 0.54 gives an example of a sequence of closed subsets of $\mathbf{R}$ whose union is the interval $(0, 1)$, which is not closed.

The empty set $\varnothing$ and the whole space $\mathbf{R}^n$ are both subsets of $\mathbf{R}^n$ that are both open and closed. The next result says that there are no other such sets.

0.65 *sets that are both open and closed*

The only subsets of $\mathbf{R}^n$ that are both open and closed are $\varnothing$ and $\mathbf{R}^n$.

Proof Suppose A is a subset of $\mathbf{R}^n$ that is both open and closed. Suppose $A \neq \varnothing$ and $A \neq \mathbf{R}^n$ (this will be a proof by contradiction). Thus there exist $a \in A$ and $b \in \mathbf{R}^n \setminus A$. Let

$$T = \{t \in [0, 1] : (1 - t)a + tb \in A\}$$

and let

$$s = \sup T.$$

The set T is nonempty because $0 \in T$; thus $s \in [0, 1]$. Let

$$c = (1 - s)a + sb.$$

Suppose $c \in A$. Then $s \neq 1$ (because otherwise $c = b \notin A$). Because $s \in T$ and A is open, T contains numbers slightly larger than s, which contradicts the definition of s as an upper bound of T.

Suppose $c \in \mathbf{R}^n \setminus A$. Then $s \neq 0$ (because otherwise $c = a \in A$). Because $s \notin T$ and $\mathbf{R}^n \setminus A$ is open, T contains no numbers slightly less than s, which contradicts the definition of s as the least upper bound of T.

Thus we arrive at a contradiction whether $c \in A$ or $c \in \mathbf{R}^n \setminus A$, completing the proof. ∎

EXERCISES D

1 Suppose $a_1, a_2, \ldots$ and $c_1, c_2, \ldots$ are convergent sequences in $\mathbf{R}^n$. Prove that

$$\lim_{k\to\infty}(a_k + c_k) = \lim_{k\to\infty} a_k + \lim_{k\to\infty} c_k.$$

2 Suppose $a_1, a_2, \ldots$ and $c_1, c_2, \ldots$ are convergent sequences in $\mathbf{R}$. Prove that

$$\lim_{k\to\infty}(a_k c_k) = \bigl(\lim_{k\to\infty} a_k\bigr)\bigl(\lim_{k\to\infty} c_k\bigr).$$

3 Prove or give a counterexample: If $a_1, a_2, \ldots$ and $c_1, c_2, \ldots$ are convergent sequences in $\mathbf{R}$ and $c_k \neq 0$ for each $k \in \mathbf{Z}^+$, then

$$\lim_{k\to\infty} \frac{a_k}{c_k} = \frac{\displaystyle\lim_{k\to\infty} a_k}{\displaystyle\lim_{k\to\infty} c_k}.$$

4 Suppose $a \in \mathbf{R}$. Prove that there exists a sequence $a_1, a_2, \ldots$ of rational numbers such that $a = \lim_{k\to\infty} a_k$.

5 Suppose $a \in \mathbf{R}$. Prove that there exists a sequence $a_1, a_2, \ldots$ of irrational numbers such that $a = \lim_{k\to\infty} a_k$.

6 Prove that the union of a sequence of countable sets is a countable set.

7 Suppose X is a set and $w\colon X \to [0, \infty)$ is a function such that

$$\sup\Bigl\{\sum_{k=1}^{n} w(x_k) : n \in \mathbf{Z}^+ \text{ and } x_1, \ldots, x_n \text{ are distinct elements of } X\Bigr\} < \infty.$$

Prove that $\{x \in X : w(x) > 0\}$ is a countable set.

8 Suppose G is an open subset of $\mathbf{R}$. Prove that $\inf G \notin G$ and $\sup G \notin G$.

9 Suppose F is a nonempty closed set of positive numbers. Prove that $\inf F \in F$.

10 Suppose G is an open subset of $\mathbf{R}^n$ and F is a closed subset of $\mathbf{R}^n$. Prove that the set difference $G \setminus F$ is an open subset of $\mathbf{R}^n$.

11 Suppose F is a closed subset of $\mathbf{R}^n$ and G is an open subset of $\mathbf{R}^n$. Prove that the set difference $F \setminus G$ is a closed subset of $\mathbf{R}^n$.

12 Suppose $a_1, a_2, \ldots$ is a convergent sequence in $\mathbf{R}^n$ with limit L. Prove that $\{L\} \cup \{a_1, a_2, \ldots\}$ is a closed subset of $\mathbf{R}^n$.

13 Suppose A is a subset of $\mathbf{R}$ that does not contain 0. Let $A^{-1} = \{a^{-1} : a \in A\}$.

(a) Prove that A is open if and only if A^{-1} is open.

(b) Give an example of a closed subset A of $\mathbf{R}$ that does not contain 0 such that A^{-1} is not a closed subset of $\mathbf{R}$.

14 Prove or give a counterexample: If G is an open subset of $\mathbf{R}$, then $\{a^2 : a \in G\}$ is open.

15 Prove or give a counterexample: If F is a closed subset of $\mathbf{R}$, then $\{a^2 : a \in F\}$ is closed.

16 Prove or give a counterexample: If F is a subset of $\mathbf{R}$ such that $\{a^2 : a \in F\}$ is closed, then F is closed.

17 Prove that a subset F of $\mathbf{R}$ is closed if and only if $F \cap [-n, n]$ is closed for every $n \in \mathbf{Z}^+$.

18 Suppose F is a closed subset of $\mathbf{R}$ and $\mathbf{Q} \subseteq F$. Prove that $F = \mathbf{R}$.

19 Suppose F is a closed subset of $\mathbf{R}$ and $(\mathbf{R} \setminus \mathbf{Q}) \subseteq F$. Prove that $F = \mathbf{R}$.

20 Prove that

$$\{a \in \mathbf{R}^n : \|a - b\|_\infty \leq \delta\} \quad \text{and} \quad \{a \in \mathbf{R}^n : \|a - b\| \leq \delta\}$$

are both closed subsets of $\mathbf{R}^n$ for every $b \in \mathbf{R}^n$ and every $\delta > 0$.

21 Prove that every closed subset of $\mathbf{R}$ is the intersection of some sequence of open subsets of $\mathbf{R}$, each of which has the form $(-\infty, a) \cup (b, \infty)$ for some $a, b \in \mathbf{R}$.

22 Suppose X is a set and A, E are subsets of X. Show that

$$A \cap E = X \setminus \big((X \setminus A) \cup (X \setminus E)\big) \quad \text{and} \quad A \cup E = X \setminus \big((X \setminus A) \cap (X \setminus E)\big).$$

23 Suppose F_1 and F_2 are disjoint closed subsets of $\mathbf{R}$ such that $F_1 \cup F_2$ is an interval. Prove that $F_1 = \varnothing$ or $F_2 = \varnothing$.

24 Suppose $G_1, G_2, \dots$ is a disjoint sequence of open sets whose union is an interval. Prove that $G_k = \varnothing$ for all $k \in \mathbf{Z}^+$ with at most one exception.

25 Construct a one-to-one function from $\mathbf{R}$ onto $\mathbf{R} \setminus \mathbf{Q}$.

26 Prove that if E is a subset of $\mathbf{R}^n$ and G is an open subset of $\mathbf{R}^n$, then $E + G$ (which is defined to be $\{x + y : x \in E, y \in G\}$) is an open subset of $\mathbf{R}^n$.

E Sequences and Continuity

Bolzano–Weierstrass Theorem

We now define some types of sequences that will turn out to be especially useful.

0.66 Definition ***increasing; decreasing; monotone***

A sequence $a_1, a_2, \ldots$ of real numbers is called

- *increasing* if $a_k \leq a_{k+1}$ for every $k \in \mathbf{Z}^+$;
- *decreasing* if $a_k \geq a_{k+1}$ for every $k \in \mathbf{Z}^+$;
- *monotone* if it is either increasing or decreasing.

0.67 Example *monotone sequences*

- The sequence $2, 4, 6, 8, \ldots$ (here the k^{th} term is $2k$) is increasing.
- The sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ (here the k^{th} term is $\frac{1}{k}$) is decreasing.
- Both of the two previous sequences are monotone.
- The sequence
$$1, \tfrac{1}{2}, 3, \tfrac{1}{4}, 5, \tfrac{1}{6}, \ldots$$
(here the k^{th} term is k if k is odd and $\frac{1}{k}$ if k is even) is neither increasing nor decreasing; thus this sequence is not monotone.

The next definition should not be a surprise.

0.68 Definition ***bounded***

- A set $A \subseteq \mathbf{R}^n$ is called *bounded* if $\sup\{\|a\|_\infty : a \in A\} < \infty$.
- A function into $\mathbf{R}^n$ is called *bounded* if its range is a bounded subset of $\mathbf{R}^n$.
- As a special case of the previous bullet point, a sequence $a_1, a_2, \ldots$ of elements of $\mathbf{R}^n$ is called *bounded* if $\sup\{\|a_k\|_\infty : k \in \mathbf{Z}^+\} < \infty$.

To test that you are comfortable with this terminology, make sure that you can show that a set $A \subseteq \mathbf{R}$ is bounded if and only if A has an upper bound and A has a lower bound.

As you should verify, if a sequence of real numbers converges, then it is bounded. The next result states that the converse is true for monotone sequences.

Note the crucial role that the completeness of the field of real numbers plays in the next proof.

0.69 *bounded monotone sequences converge*

Every bounded monotone sequence of real numbers converges.

Proof Suppose $a_1, a_2, \dots$ is a bounded monotone sequence of real numbers.

First we consider the case where $a_1, a_2, \dots$ is an increasing sequence. Let

$$L = \sup\{a_1, a_2, \dots\}.$$

Suppose $\varepsilon > 0$. Then $L - \varepsilon$ is not an upper bound of the sequence. Thus there exists a positive integer m such that $L - \varepsilon < a_m$. Because the sequence is increasing, we conclude that $L - \varepsilon < a_k$ for all integers $k \geq m$. Thus $|L - a_k| = L - a_k < \varepsilon$ for all integers $k \geq m$. Hence $\lim_{k\to\infty} a_k = L$, completing the proof in this case.

Now consider the case where $a_1, a_2, \dots$ is a bounded decreasing sequence. Then the sequence $-a_1, -a_2, \dots$ is a bounded increasing sequence, which converges by the first case that we considered. Multiplying by -1, we conclude that $a_1, a_2, \dots$ converges, completing the proof. ■

0.70 Definition *subsequence*

A *subsequence* of a sequence $a_1, a_2, a_3, \dots$ is a sequence of the form

$$a_{k_1}, a_{k_2}, a_{k_3}, \dots,$$

where $k_1, k_2, k_3, \dots$ are positive integers with $k_1 < k_2 < k_3 < \cdots$.

0.71 Example *subsequence*

The sequence $2, 5, 8, 11, \dots$ (here the k^{th} term is $3k - 1$) has $2, 11, 26, 47, \dots$ as a subsequence (here $k_j = j^2$).

The next result provides us with a powerful tool.

0.72 *monotone subsequence*

Every sequence of real numbers has a monotone subsequence.

Proof Suppose $a_1, a_2, \dots$ is a sequence of real numbers. For $m \in \mathbf{Z}^+$, we say that m is a *peak* of this sequence if $a_m \geq a_k$ for all $k > m$.

First consider the case where the sequence $a_1, a_2, \dots$ has an infinite number of peaks. Thus there exist positive integers $m_1 < m_2 < m_3 < \cdots$ such that m_j is a peak of this sequence for each $j \in \mathbf{Z}^+$. Because m_j is a peak, we have $a_{m_j} \geq a_{m_{j+1}}$ for each $j \in \mathbf{Z}^+$. Thus the subsequence $a_{m_1}, a_{m_2}, a_{m_3}, \dots$ is decreasing.

Now consider the case where the sequence $a_1, a_2, \dots$ has only a finite number of peaks. Thus there exists $m_1 \in \mathbf{Z}^+$ such that k is not a peak for every $k \geq m_1$. Because m_1 is not a peak, there exists $m_2 > m_1$ such that $a_{m_1} < a_{m_2}$. Because m_2 is not a peak, there exists $m_3 > m_2$ such that $a_{m_2} < a_{m_3}$, and so on. Thus the subsequence $a_{m_1}, a_{m_2}, a_{m_3}, \dots$ is increasing. ■

Now we come to a major theorem that has multiple important consequences. Because of the results we have already proved about monotone sequences, the proof below is quite short.

0.73 *Bolzano–Weierstrass Theorem*

Every bounded sequence in $\mathbf{R}^n$ has a convergent subsequence.

Proof Suppose $b_1, b_2, \ldots$ is a bounded sequence in $\mathbf{R}^n$.

For $n = 1$, the desired result follows immediately from combining two results: every sequence of real numbers has a monotone subsequence (0.72), and every bounded monotone sequence converges (0.69).

For $n > 1$, first take a convergent subsequence of the sequence of first coordinates of $b_1, b_2, \ldots$. Then take a convergent subsequence of second coordinates of that subsequence. Continue this process until taking a convergent subsequence of n^{th} coordinates. The result on coordinatewise limits (0.48) now gives a convergent subsequence of $b_1, b_2, \ldots$, as desired. ∎

Plaque honoring Bernard Bolzano (1781–1848) in his native city Prague. Bolzano proved the result above in 1817. However, this work did not become widely known in the international mathematical community until the work of Karl Weierstrass (1815–1897) about a half-century later.

The next result is called a characterization of closed bounded sets because the converse, although less important, is also true (see Exercise 3 in this section).

0.74 *characterization of closed bounded sets*

Suppose F is a closed bounded subset of $\mathbf{R}^n$. Then every sequence of elements of F has a subsequence that converges to an element of F.

Proof Consider a sequence of elements of F. Because F is a bounded set, this sequence is bounded and thus has a convergent subsequence (by the Bolzano–Weierstrass Theorem, 0.73). Because F is closed, the limit of this convergent subsequence is in F (by 0.62). ∎

Continuity and Uniform Continuity

Although Isaac Newton (1643–1727) and Gottfried Leibniz (1646–1716) invented calculus in the seventeenth century, a rigorous definition of continuity did not arise until the nineteenth century.

0.75 Definition ***continuity***

Suppose $A \subseteq \mathbf{R}^m$ and $f: A \to \mathbf{R}^n$ is a function.

- For $b \in A$, the function f is called *continuous* at b if for every $\varepsilon > 0$, there exists $\delta > 0$ such that
$$\|f(a) - f(b)\|_\infty < \varepsilon$$
for all $a \in A$ with $\|a - b\|_\infty < \delta$.
- The function f is called *continuous* if f is continuous at b for every $b \in A$.

In the first bullet point above, continuity is defined at an element of the domain. The second bullet point above establishes the convention that simply calling a function continuous means that the function is continuous at every element of its domain.

The next result, whose proof is left as an exercise, allows us to think about continuity in terms of limits of sequences.

0.76 ***continuity via sequences***

Suppose $A \subseteq \mathbf{R}^m$ and $f: A \to \mathbf{R}^n$ is a function. Suppose $b \in A$. Then f is continuous at b if and only if

$$\lim_{k\to\infty} f(b_k) = f(b)$$

for every sequence $b_1, b_2, \dots$ in A such that $\lim\limits_{k\to\infty} b_k = b$.

The concept of uniform continuity also evolved in the nineteenth century.

0.77 Definition ***uniform continuity***

Suppose $A \subseteq \mathbf{R}^m$. A function $f: A \to \mathbf{R}^n$ is called *uniformly continuous* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|f(a) - f(b)\|_\infty < \varepsilon$$

for all $a, b \in A$ with $\|a - b\|_\infty < \delta$.

The symbols appearing in the definition of continuity and in the definition of uniform continuity are the same, but pay careful attention to the different order of the quantifiers. In the definition above of continuity, δ can depend upon b, but in the definition of uniform continuity, δ cannot depend upon b.

Clearly every uniformly continuous function is continuous, but the converse is not true, as shown by the following example.

0.78 Example *a continuous function that is not uniformly continuous*

Define $f\colon \mathbf{R} \to \mathbf{R}$ by $f(x) = x^2$. Then

$$\begin{aligned} f(n + \tfrac{1}{n}) - f(n) &= 2 + \tfrac{1}{n^2} \\ &> 2 \end{aligned}$$

for every $n \in \mathbf{Z}^+$. The inequality above implies that f is not uniformly continuous.

The following remarkable result states that for functions whose domain is a closed bounded subset of $\mathbf{R}^m$, continuity implies uniform continuity. This result plays a crucial role in showing that continuous functions are Riemann integrable (see 1.11 in *Measure, Integration & Real Analysis*).

0.79 *continuity implies uniform continuity on closed bounded sets*

Every continuous $\mathbf{R}^n$-valued function on each closed bounded subset of $\mathbf{R}^m$ is uniformly continuous.

Proof Suppose F is a closed bounded subset of $\mathbf{R}^m$ and $g\colon F \to \mathbf{R}^n$ is continuous. We want to show that g is uniformly continuous.

Suppose g is not uniformly continuous. Then there exists $\varepsilon > 0$ such that for each $k \in \mathbf{Z}^+$, there exist $a_k, b_k \in F$ with

$$\|a_k - b_k\|_\infty < \tfrac{1}{k} \quad \text{and} \quad \|g(a_k) - g(b_k)\|_\infty \geq \varepsilon.$$

Because F is bounded, the sequence $a_1, a_2, \dots$ is bounded. Thus by the Bolzano–Weierstrass Theorem (0.73), some subsequence $a_{k_1}, a_{k_2}, \dots$ converges to some limit a. Because F is closed, we have $a \in F$ (by 0.62).

Now

$$\begin{aligned} \|a - b_{k_j}\|_\infty &= \|(a - a_{k_j}) + (a_{k_j} - b_{k_j})\|_\infty \\ &\leq \|a - a_{k_j}\|_\infty + \|a_{k_j} - b_{k_j}\|_\infty \\ &< \|a - a_{k_j}\|_\infty + \tfrac{1}{k_j}, \end{aligned}$$

which implies that $\lim_{j\to\infty} b_{k_j} = a$.

Because g is continuous at a and $\lim_{j\to\infty} a_{k_j} = a$ and $\lim_{j\to\infty} b_{k_j} = a$, we conclude that

$$\lim_{j\to\infty} g(a_{k_j}) = g(a) \quad \text{and} \quad \lim_{j\to\infty} g(b_{k_j}) = g(a).$$

Thus

$$\lim_{j\to\infty} \bigl(g(a_{k_j}) - g(b_{k_j})\bigr) = 0.$$

The equation above contradicts the inequality $\|g(a_k) - g(b_k)\|_\infty \geq \varepsilon$, which holds for all $k \in \mathbf{Z}^+$. This contradiction means that our assumption that g is not uniformly continuous is false, completing the proof. ■

Max and Min on Closed Bounded Subsets of $\mathbf{R}^n$

If $f\colon (2,3) \to \mathbf{R}$ is the function defined by $f(x) = x^2$, then

$$\sup\{f(x) : x \in (2,3)\} = 9 \quad \text{and} \quad \inf\{f(x) : x \in (2,3)\} = 4.$$

However, there is no x in the domain of f such that $f(x) = 9$ or $f(x) = 4$. In other words, this function f attains neither a maximum value nor a minimum value. The next result shows that such behavior cannot happen for continuous functions on closed bounded subsets of $\mathbf{R}^m$.

0.80 ***maximum and minimum attained on closed bounded sets***

Every continuous real-valued function on each closed bounded subset of $\mathbf{R}^m$ attains its maximum and minimum.

Proof Suppose F is a nonempty closed bounded subset of $\mathbf{R}^m$ and $g\colon F \to \mathbf{R}$ is a continuous function. Let $a_1, a_2, \ldots$ be a sequence in F such that

$$\lim_{k\to\infty} g(a_k) = \sup\{g(x) : x \in F\}.$$

Some subsequence of $a_1, a_2, \ldots$ converges to some $a \in F$ (by 0.74). Because g is continuous at a, the equation above implies that

$$g(a) = \sup\{g(x) : x \in F\}.$$

Thus $\sup\{g(x) : x \in F\} < \infty$ and g attains its maximum on F, as desired.

To prove that g also attains its minimum on F, use similar ideas or apply the result about a maximum to the function $-g$. ■

0.81 Definition ***image***

Suppose S, T are sets and $g\colon S \to T$ is a function. If $A \subseteq S$, then the *image* of A under g, denoted $g(A)$, is the subset of T defined by

$$g(A) = \{g(x) : x \in A\}.$$

For example, if $g\colon \mathbf{R} \to \mathbf{R}$ is the function defined by $g(x) = \cos x$, then $g([\frac{\pi}{2}, \pi]) = [-1, 0]$.

In the next proof, convergent subsequences will again play a key role.

0.82 ***continuous image of a closed bounded set is closed and bounded***

Suppose F is a closed bounded subset of $\mathbf{R}^m$ and $g\colon F \to \mathbf{R}^n$ is continuous. Then $g(F)$ is a closed bounded subset of $\mathbf{R}^n$.

Proof By 0.80, $g(F)$ is bounded.

To prove that $g(F)$ is closed, suppose $g(a_1), g(a_2), \ldots$ is a convergent sequence, where each $a_k \in F$. Let $t = \lim_{k\to\infty} g(a_k)$. By 0.74, some subsequence of $a_1, a_2, \ldots$ converges to some $a \in F$. Because g is continuous at a, this implies that $t = g(a)$. Thus $t \in g(F)$. By 0.62, this implies that $g(F)$ is closed, as desired. ■

EXERCISES E

1 Prove that every convergent sequence of elements of $\mathbf{R}^n$ is bounded.

2 Prove that a sequence of elements of $\mathbf{R}^n$ converges if and only if every subsequence of the sequence converges.

3 Prove the converse of 0.74. Specifically, prove that if F is a subset of $\mathbf{R}^n$ with the property that every sequence of elements of F has a subsequence that converges to an element of F, then F is closed and bounded.

4 Define $f\colon \mathbf{R} \to \mathbf{R}$ as follows:

$$f(a) = \begin{cases} 0 & \text{if } a \text{ is irrational,} \\ \frac{1}{n} & \text{if } a \text{ is rational and } n \text{ is the smallest positive integer} \\ & \text{such that } a = \frac{m}{n} \text{ for some integer } m. \end{cases}$$

At which numbers in $\mathbf{R}$ is f continuous?

5 Prove 0.76, which characterizes continuity via sequences.

6 Show that the function $f\colon (0,\infty) \to \mathbf{R}$ defined by $f(x) = \frac{1}{x}$ is not uniformly continuous.

7 Suppose $p \in (0,\infty)$. Show that the function $f\colon \mathbf{R} \to \mathbf{R}$ defined by $f(x) = |x|^p$ is uniformly continuous if and only if $p \in (0,1]$.

8 Prove or give a counterexample: If $f\colon \mathbf{R} \to \mathbf{R}$ is a bounded continuous function, then f is uniformly continuous.

9 Prove or give a counterexample: If $f\colon (0,1) \to \mathbf{R}$ is a bounded continuous function, then f is uniformly continuous.

10 Prove that if A is a bounded subset of $\mathbf{R}^m$ and $f\colon A \to \mathbf{R}$ is uniformly continuous, then f is a bounded function.

11 Prove that if $f\colon \mathbf{R} \to \mathbf{R}$ is differentiable everywhere and f' is a bounded function on $\mathbf{R}$, then f is uniformly continuous.

12 Give an example of a uniformly continuous function $f\colon [-1,1] \to \mathbf{R}$ such that f is differentiable at every element of $[-1,1]$ but f' is not a bounded function on $[-1,1]$.

13 Prove or give a counterexample: If $f\colon \mathbf{R}^m \to \mathbf{R}^n$ is continuous and

$$\|f(x)\| < \frac{1}{\|x\|}$$

for all $x \in \mathbf{R}^m$ with $\|x\| > 1$, then f is uniformly continuous.

14 Prove or give a counterexample: The sum of two uniformly continuous functions from $\mathbf{R}^m$ to $\mathbf{R}^n$ is uniformly continuous.

15 Prove or give a counterexample: The product of two uniformly continuous functions from $\mathbf{R}$ to $\mathbf{R}$ is uniformly continuous.

16 Prove or give a counterexample: If $f\colon \mathbf{R} \to (0,\infty)$ is uniformly continuous, then the function $\frac{1}{f}$ is uniformly continuous on $\mathbf{R}$.

17 Prove or give a counterexample: If $f, g\colon \mathbf{R} \to \mathbf{R}$ are uniformly continuous functions, then the composition $f \circ g\colon \mathbf{R} \to \mathbf{R}$ is uniformly continuous.

If S, T are sets and $f\colon S \to T$ is a function and $A \subseteq T$, then the set $f^{-1}(A)$ is defined by

$$f^{-1}(A) = \{s \in S : f(s) \in A\}.$$

18 Suppose $h\colon \mathbf{R}^m \to \mathbf{R}^n$ is a function. Prove that h is continuous if and only if $h^{-1}(G)$ is an open subset of $\mathbf{R}^m$ for every open subset G of $\mathbf{R}^n$.

19 Suppose $h\colon \mathbf{R}^m \to \mathbf{R}^n$ is a function. Prove that h is continuous if and only if $h^{-1}(F)$ is a closed subset of $\mathbf{R}^m$ for every closed subset F of $\mathbf{R}^n$.

20 Give an example of a decreasing sequence $G_1 \supseteq G_2 \supseteq \cdots$ of nonempty open bounded subsets of $\mathbf{R}$ such that $\bigcap_{k=1}^{\infty} G_k = \varnothing$.

21 Give an example of a decreasing sequence $F_1 \supseteq F_2 \supseteq \cdots$ of nonempty closed subsets of $\mathbf{R}$ such that $\bigcap_{k=1}^{\infty} F_k = \varnothing$.

22 Suppose $F_1 \supseteq F_2 \supseteq \cdots$ is a decreasing sequence of nonempty closed bounded subsets of $\mathbf{R}^n$. Prove that $\bigcap_{k=1}^{\infty} F_k \neq \varnothing$.

23 Prove that every continuous real-valued function on each closed subset of $\mathbf{R}$ can be extended to a continuous real-valued function on $\mathbf{R}$. More precisely, prove that if F is a closed subset of $\mathbf{R}$ and $g\colon F \to \mathbf{R}$ is continuous, then there exists a continuous function $h\colon \mathbf{R} \to \mathbf{R}$ such that $g(x) = h(x)$ for all $x \in F$.

24 Prove or give a counterexample: If G is a bounded open subset of $\mathbf{R}$ and $h\colon G \to \mathbf{R}$ is continuous, then $h(G)$ is an open subset of $\mathbf{R}$.

25 Prove or give a counterexample: If F is a closed subset of $\mathbf{R}$ and $h\colon F \to \mathbf{R}$ is continuous, then $h(F)$ is a closed subset of $\mathbf{R}$.

26 Suppose F is a subset of $\mathbf{R}^n$ such that every continuous real-valued function on F attains a maximum. Prove that F is closed and bounded.

27 Suppose $f\colon \mathbf{R} \to \mathbf{R}$ is an increasing function [meaning that $a < b$ implies $f(a) \leq f(b)$]. Prove that there exists a countable set $A \subseteq \mathbf{R}$ such that f is continuous at each element of $\mathbf{R} \setminus A$.

28 Suppose $a_1, a_2, \dots$ is a sequence of real numbers. For each $k \in \mathbf{Z}^+$, define $g_k\colon \mathbf{R} \to \mathbf{R}$ by

$$g_k(x) = \begin{cases} 0 & \text{if } a_k \geq x, \\ 1 & \text{if } a_k < x. \end{cases}$$

Define $f\colon \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \sum_{k=1}^{\infty} \frac{g_k(x)}{2^k}.$$

Suppose $x \in \mathbf{R}$. Prove that f is continuous at x if and only if $x \notin \{a_1, a_2, \dots\}$.

29 Prove that the continuous image of an interval is an interval. In other words, prove that if $f\colon [a, b] \to \mathbf{R}$ is continuous and t is between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ such that $f(c) = t$.
[*This result is called the Intermediate Value Theorem.*]

30 Prove that every continuous function from $\mathbf{R}$ to $\mathbf{R} \setminus \mathbf{Q}$ is a constant function.

31 Prove that every polynomial with odd degree has a real zero. In other words, prove that if $p\colon \mathbf{R} \to \mathbf{R}$ is a polynomial with odd degree, then there exists $b \in \mathbf{R}$ such that $p(b) = 0$.

A sequence $a_1, a_2, \dots$ of elements of $\mathbf{R}^n$ is called a Cauchy sequence if for every $\varepsilon > 0$, there exists $m \in \mathbf{Z}^+$ such that $|a_j - a_k| < \varepsilon$ for all integers j and k greater than m.

32 Prove that every convergent sequence of elements of $\mathbf{R}^n$ is a Cauchy sequence.

33 (a) Prove that every Cauchy sequence of elements of $\mathbf{R}^n$ is bounded.

(b) Prove that if some subsequence of a Cauchy sequence of elements of $\mathbf{R}^n$ converges to some $L \in \mathbf{R}^n$, then the Cauchy sequence has limit L.

(c) Prove that every Cauchy sequence of elements of $\mathbf{R}^n$ converges.

34 Prove that if F_1 is a closed subset of $\mathbf{R}^n$ and F_2 is a closed bounded subset of $\mathbf{R}^n$, then $F_1 + F_2$ (which is defined to be $\{x + y : x \in F_1, y \in F_2\}$) is closed.

35 Give an example of two closed subsets of $\mathbf{R}$ whose sum is not closed.

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